Example 2. *Find the inverse Laplace transform of the function* $\dfrac{1}{s^5+s^3}$.

Sol. $$\frac{1}{s^5+s^3}=\frac{1}{s^3(s^2+1)}=\frac{1}{s^3}\left(\frac{1}{s^2+1}\right)=\frac{1}{s^3}\,F(s), \text{ where } F(s)=\frac{1}{s^2+1}.$$

$$L^{-1}(F(s)) = L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t = f(t), \text{ say}$$

$\therefore$ By the **property of Laplace transform of integrals,**

$$L^{-1}\left(\frac{1}{s}F(s)\right)=\int_0^t F(T)\,dT=\int_0^t \sin T\,dT=-\cos T\Big|_0^t = -\cos t+\cos 0 = 1-\cos t.$$

Again by using the same formula,

$$L^{-1}\left(\frac{1}{s}\cdot\frac{1}{s}F(s)\right)=\int_0^t (1-\cos T)\,dT=(T-\sin T)\Big|_0^t = t-\sin t.$$

$$\therefore \qquad L^{-1}\left(\frac{1}{s^2}F(s)\right) = t-\sin t.$$

Again by using the same formula,

$$L^{-1}\left(\frac{1}{s}\cdot\frac{1}{s^2}F(s)\right)=\int_0^t (T-\sin T)\,dT=\left(\frac{T^2}{2}+\cos T\right)\Bigg|_0^t = \frac{t^2}{2}+\cos t-1$$

$$\therefore \qquad L^{-1}\left(\frac{1}{s^3}F(s)\right)=\frac{t^2}{2}+\cos t-1$$

or $$L^{-1}\left(\frac{1}{s^5+s^3}\right)=\boldsymbol{\frac{t^2}{2}+\cos t-1.}$$

Remark. The above problem can also be solved by the "method of partial fractions".

2.10. VALUE OF L^{-1} (F′(s)) IN TERMS OF L^{-1}(F(s))

By the property of **differentiation of Laplace transforms**

if $\qquad L(f(t)) = F(s)$, then $L(t\,f(t)) = -F'(s)$.

$\therefore$ If $\quad L^{-1}(F(s)) = f(t)$, then $F'(s) = L(-t\,f(t))$.

$\therefore$ **If $\quad L^{-1}(F(s)) = f(t)$, then $L^{-1}(F'(s)) = -t\,f(t)$.**

Example 3. *Find the inverse Laplace transform of the function* $\dfrac{s}{(s^2+4)^2}$.

Sol. $$\int \frac{s}{(s^2+4)^2}\,ds=\frac{1}{2}\int\frac{2s}{(s^2+4)^2}\,ds=\frac{1}{2}\cdot\frac{(s^2+4)^{-1}}{-1}=-\frac{1}{2(s^2+4)}=F(s), \text{ say.}$$

$$\therefore \qquad F'(s)=\frac{s}{(s^2+4)^2}$$

Also $$L^{-1}(F(s))=L^{-1}\left(-\frac{1}{2(s^2+4)}\right)=-\frac{1}{4}L^{-1}\left(\frac{2}{s^2+2^2}\right)=-\frac{1}{4}\sin 2t=f(t), \text{ say.}$$

$\therefore$ By the **property of differentiation of Laplace transforms,** we have

$$L^{-1}(F'(s)) = -t f(t).$$

$$\therefore \quad L^{-1}\left(\frac{s}{(s^2+4)^2}\right) = -t\left(-\frac{1}{4}\sin 2t\right) = \frac{1}{4}t\sin 2t$$

$$\therefore \quad L^{-1}\left(\frac{s}{(s^2+4)^2}\right) = \mathbf{\frac{1}{4}\, t \sin 2t.}$$

Example 4. *Find the inverse Laplace transform of the function* $\dfrac{2s+6}{(s^2+6s+10)^2}$.

Sol. $$L^{-1}\left[\frac{2s+6}{(s^2+6s+10)^2}\right] = L^{-1}\left[\frac{2(s+3)}{((s+3)^2+1)^2}\right] = e^{-3t}L^{-1}\left[\frac{2s}{(s^2+1)^2}\right] \quad ...(1)$$

Now, we require the value of $L^{-1}\left[\dfrac{2s}{(s^2+1)^2}\right]$.

$$\int \frac{2s}{(s^2+1)^2}ds = \frac{(s^2+1)^{-1}}{-1} = -\frac{1}{s^2+1} = F(s), \quad \text{say.}$$

$$\therefore \quad F'(s) = \frac{2s}{(s^2+1)^2}$$

Also $$L^{-1}(F(s)) = L^{-1}\left(-\frac{1}{s^2+1}\right) = -\sin t = f(t), \quad \text{say.}$$

$\therefore$ By the **property of differentiation of Laplace transforms,** we have

$$L^{-1}(F'(s)) = -t f(t).$$

$$\Rightarrow \quad L^{-1}\left(\frac{2s}{(s^2+1)^2}\right) = -t(-\sin t) \Rightarrow L^{-1}\left(\frac{2s}{(s^2+1)^2}\right) = t\sin t.$$

$$\therefore \ (1) \Rightarrow L^{-1}\left(\frac{2s+6}{(s^2+6s+10)^2}\right) = \mathbf{e^{-3t}\, t \sin t}.$$

Example 5. *Find the inverse Laplace transform of the function* $\dfrac{1}{(s^2+a^2)^2}$.

Sol. Let $$F(s) = \frac{1}{s^2+a^2}.$$

$$\therefore \quad L^{-1}(F(s)) = L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}L^{-1}\left(\frac{a}{s^2+a^2}\right) = \frac{1}{a}\sin at = f(t), \text{ say.}$$

Using, $L^{-1}(F'(s)) = -t f(t)$, we get

$$L^{-1}\left(\left(\frac{1}{s^2+a^2}\right)'\right) = -t \cdot \frac{1}{a}\sin at.$$

$$\Rightarrow \qquad L^{-1}\left(\frac{-2s}{(s^2+a^2)^2}\right) = -\frac{1}{a}t\sin at$$

$$\Rightarrow \qquad L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = \frac{1}{2a}t\sin at = g(t), \quad \text{say}.$$

Using formula for **Laplace transform of integrals,** we get

$$L^{-1}\left(\frac{1}{s}\cdot\frac{s}{(s^2+a^2)^2}\right) = \int_0^t g(T)\,dT\,.$$

$$\Rightarrow \qquad L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right) = \int_0^t \frac{1}{2a}T\sin aT\,dT = \frac{1}{2a}\left[-\frac{T\cos aT}{a}+\frac{\sin aT}{a^2}\right]_0^t$$

$$= \frac{1}{2a^3}[-at\cos at+\sin at]$$

$$\therefore \qquad L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right) = \frac{\mathbf{1}}{\mathbf{2a^3}}\,\mathbf{[\sin at - at\cos at]}.$$

2.11. VALUE OF $L^{-1}\left(\int_s^\infty F(s)\,dS\right)$ IN TERMS OF L^{-1} (F(s))

By the property of **integration of Laplace transforms**

if $\qquad L(f(t)) = F(s)$, then $L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(S)\,dS.$

$\therefore$ **If** $\qquad \mathbf{L^{-1}(F(s)) = f(t),\ then\ L^{-1}\left(\int_s^\infty F(S)\,dS\right) = \frac{f(t)}{t}}$**.**

Example 6. *Find the inverse Laplace transform of the function log* $\left(1+\frac{a^2}{s^2}\right)$.

Sol. $\frac{d}{ds}\left(\log\left(1+\frac{a^2}{s^2}\right)\right) = \frac{d}{ds}\left(\log(s^2+a^2) - 2\log s\right) = \frac{2s}{s^2+a^2} - \frac{2}{s} = F(s),\quad$ say.

$$\therefore \qquad \int F(s)\,ds = \log\left(1+\frac{a^2}{s^2}\right)$$

Also, $\qquad L^{-1}(F(s)) = L^{-1}\left(\frac{2s}{s^2+a^2}-\frac{2}{s}\right) = 2L^{-1}\left(\frac{s}{s^2+a^2}\right) - 2L^{-1}\left(\frac{1}{s}\right)$

$$= 2\cos at - 2(1) = 2\cos at - 2 = f(t), \quad \text{say}.$$

$\therefore$ By the **property of integration of Laplace transforms**, we have

$$L^{-1}\left(\int_s^\infty F(S)\,dS\right) = \frac{f(t)}{t}.$$

$$\therefore \qquad L^{-1}\left(\log\left(1+\frac{a^2}{S^2}\right)\Bigg|_s^{\infty}\right) = \frac{2\cos at - 2}{t}$$

$$\therefore \qquad L^{-1}\left(0-\log\left(1+\frac{a^2}{s^2}\right)\right) = \frac{2\cos at - 2}{t}$$

$$\therefore \qquad L^{-1}\left(\log\left(1+\frac{a^2}{s^2}\right)\right) = \mathbf{\frac{2-2\cos at}{t}}.$$

Example 7. *Find the inverse Laplace transform of the function log* $\dfrac{s^2+1}{(s-1)^2}$.

Sol. $\dfrac{d}{ds}\left(\log\dfrac{s^2+1}{(s-1)^2}\right) = \dfrac{d}{ds}(\log(s^2+1) - 2\log(s-1)) = \dfrac{2s}{s^2+1} - \dfrac{2}{s-1} = F(s)$, say.

$$\therefore \qquad \int F(s)\, ds = \log\frac{s^2+1}{(s-1)^2}$$

Also $\quad L^{-1}(F(s)) = L^{-1}\left(\dfrac{2s}{s^2+1} - \dfrac{2}{s-1}\right) = 2L^{-1}\left(\dfrac{s}{s^2+1}\right) - 2L^{-1}\left(\dfrac{1}{s-1}\right)$

$= 2\cos t - 2e^t = f(t)$, say.

By the **property of integration of Laplace transforms**, we have

$$L^{-1}\left(\int_s^{\infty} F(S)\, dS\right) = \frac{f(t)}{t}.$$

$$\therefore \qquad L^{-1}\left(\left(\log\frac{S^2+1}{(S-1)^2}\right)\Bigg|_s^{\infty}\right) = \frac{2\cos t - 2e^t}{t}$$

$$\therefore \qquad L^{-1}\left(0-\log\frac{s^2+1}{(s-1)^2}\right) = -\frac{2e^t - 2\cos t}{t}$$

$$\therefore \qquad L^{-1}\left(\log\frac{s^2+1}{(s-1)^2}\right) = \mathbf{\frac{2e^t - 2\cos t}{t}}.$$

Example 8. *Find the inverse Laplace transform of the function* $\cot^{-1}(s+1)$.

Sol. $\dfrac{d}{ds}(\cot^{-1}(s+1)) = \dfrac{-1}{(s+1)^2+1}\,.\,1 = -\dfrac{1}{(s+1)^2+1} = F(s)$, say.

$$\therefore \qquad \int F(s)\, ds = \cot^{-1}(s+1)$$

Also, $$L^{-1}(F(s)) = L^{-1}\left(-\frac{1}{(s+1)^2+1}\right) = -L^{-1}\left(\frac{1}{(s+1)^2+1}\right)$$

$$= -e^{-t} L^{-1}\frac{1}{s^2+1} = -e^{-t}\sin t = f(t), \quad \text{say.}$$

$\therefore$ By the **property of integration of Laplace transforms**, we have

$$L^{-1}\left(\int_s^\infty F(S)\,dS\right) = \frac{f(t)}{t}.$$

$\therefore$ $$L^{-1}\left(\cot^{-1}(S+1)\Big|_s^\infty\right) = -\frac{e^{-t}\sin t}{t}$$

$\Rightarrow$ $$L^{-1}(0-\cot^{-1}(s+1)) = -\frac{e^{-t}\sin t}{t}$$

$\therefore$ $$L^{-1}(\cot^{-1}(s+1)) = \boldsymbol{\frac{e^{-t}\sin t}{t}}.$$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. If $L^{-1}(F(s)) = f(t)$, then $L^{-1}\left(\frac{1}{s}F(s)\right) = \int_0^t f(T)\,dT$.

Rule II. If $L^{-1}(F(s)) = f(t)$, then $L^{-1}(F'(s)) = -t\,f(t)$.

Rule III. If $L^{-1}(F(s)) = f(t)$, then $L^{-1}\left(\int_s^\infty F(S)dS\right) = \frac{f(t)}{t}$.

TEST YOUR KNOWLEDGE

Find the inverse Laplace transform of the following functions :

1. (*i*) $\frac{1}{s^2+4s}$ (*ii*) $\frac{1}{s(s^2+16)}$ **2.** (*i*) $\frac{1}{s^3-s}$ (*ii*) $\frac{4}{s^3-2s^2}$

3. (*i*) $\frac{1}{s(s+a)^3}$ (*ii*) $\frac{1}{s(s^2+2s+2)}$ **4.** (*i*) $\frac{7}{s^4-s^2}$ (*ii*) $\frac{\pi^5}{s^4(s^2+\pi^2)}$

5. (*i*) $\frac{s}{(s^2+a^2)^2}$ (*ii*) $\frac{s}{(s^2-9)^2}$ **6.** (*i*) $\frac{s^2-\pi^2}{(s^2+\pi^2)^2}$ (*ii*) $\frac{s+1}{(s^2+2s+2)^2}$

7. (*i*) $\log\frac{s+a}{s+b}$ (*ii*) $\log\frac{s+1}{s-1}$ **8.** (*i*) $\log\left(1-\frac{a^2}{s^2}\right)$ (*ii*) $\log\frac{1+s}{s}$

9. (*i*) $\frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$ (*ii*) $\log\frac{s^2+1}{s(s+1)}$ **10.** (*i*) $\log\frac{(s+1)^2}{(s+2)(s+3)}$ (*ii*) $\log\frac{s^2+1}{(s-1)^2}$

11. (*i*) $\cot^{-1}\frac{s}{\pi}$ (*ii*) $\tan^{-1}\frac{2}{s}$.

Answers

1. (*i*) $\frac{1}{4}-\frac{e^{-4t}}{4}$ (*ii*) $\frac{1}{16}-\frac{\cos 4t}{16}$ **2.** (*i*) $\cosh t - 1$ (*ii*) $e^{2t} - 2t - 1$

3. (*i*) $\frac{1}{a^3}-\frac{e^{-at}}{a^3}\left(\frac{a^2}{2}t^2+at+1\right)$ (*ii*) $\frac{1}{2}\left[1-(\sin t+\cos t)\,e^{-t}\right]$

4. (*i*) $7(\sinh t - t)$ (*ii*) $\sin \pi t + \frac{\pi^3}{6}t^3 - \pi t$

5. (*i*) $\frac{1}{2a}t\sin at$ (*ii*) $\frac{1}{6}t\sinh 3t$ **6.** (*i*) $t\cos \pi t$ (*ii*) $\frac{1}{2}e^{-t}\,t\sin t$

7. (*i*) $\frac{e^{-bt}-e^{-at}}{t}$ (*ii*) $\frac{e^{t}-e^{-t}}{t}$ **8.** (*i*) $\frac{2(1-\cos at)}{t}$ (*ii*) $\frac{1-e^{-t}}{t}$

9. (*i*) $\frac{\cos at-\cos bt}{t}$ (*ii*) $\frac{1+e^{-t}-2\cos t}{t}$

10. (*i*) $\frac{e^{-2t}+e^{-3t}-2e^{-t}}{t}$ (*ii*) $\frac{2(e^{t}-\cos t)}{t}$

11. (*i*) $\frac{\sin \pi t}{t}$ (*ii*) $\frac{\sin 2t}{t}$.

Hint

6. (*i*) Use $\left(\frac{s}{s^2+\pi^2}\right)' = -\frac{s^2-\pi^2}{(s^2+\pi^2)^2}$.

2.12. CONVOLUTION THEOREM

The convolution theorem is used to find the inverse Laplace transform of the product of two functions with known inverse Laplace transforms of the factors of the product. Let F(s) and G(s) be two functions with known inverse Laplace transforms $f(t)$ and $g(t)$ respectively. The convolution theorem would help us to find the inverse Laplace transform of the product F(s) G(s). In this theorem, we shall prove that the inverse Laplace transformation of the product F(s) G(s) is given by the function $\int_0^t f(\mathrm{T})\,g(t-T)\,d\mathrm{T}$ of t. This integral is called the **convolution** of the functions f and g and it is denoted by $f * g$.

$$\therefore \quad (f * g)(t) = \int_0^t f(\mathrm{T})\,g(t-T)\,d\mathrm{T}.$$

Let $z = t - \mathrm{T}$. $\therefore$ $dz = -\,d\mathrm{T}$

$$\therefore \quad (f * g)(t) = \int_t^0 f(t-z)\,g(z)\,.-dz = -\int_t^0 g(z)\,f(t-z)\,dz$$

$$= \int_0^t g(z)\,f(t-z)\,dz = \int_0^t g(\mathrm{T})\,f(t-\mathrm{T})\,d\mathrm{T} \qquad \text{(Replacing } z \text{ by T)}$$

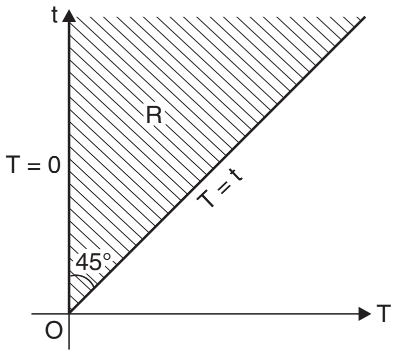
$$= (g * f)(t)$$

$$\therefore \quad \mathbf{f * g = g * f}$$

Statement. *Let f(t) and g(t) be functions such that their Laplace transforms exist. Prove that*

$$L(f * g) = L(f)\, L(g).$$

Proof. It can be proved mathematically that the Laplace transform of the convolution of f and g (*i.e.*, $f * g$) exists.



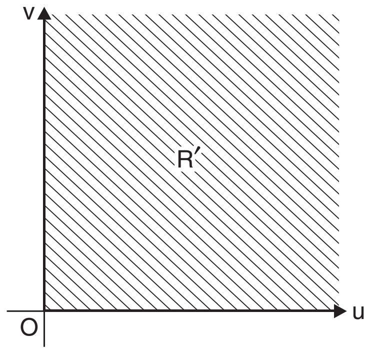
By definition, $L(f * g) = \int_0^\infty e^{-st}\,(f * g)(t)\,dt$

$$\therefore \quad L(f * g) = \int_0^\infty e^{-st}\left[\int_0^t f(T)\,g(t-T)\,dT\right]dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(T)\,g(t-T)\,dT\,dt$$

$$= \iint_R e^{-st} f(T)\,g(t-T)\,dT\,dt,$$

where R is the 45° wedge bounded by the lines T = 0 and T = t in the Tt-plane.



Let $\quad u = T$ and $v = t - T$.

$\therefore \quad T = u$ and $t = u + v$

$$\therefore \quad J = \begin{vmatrix} \frac{\partial T}{\partial u} & \frac{\partial T}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

New variables transform the region R into the region R′ in the uv-plane defined by $u \geq 0$, $v \geq 0$.

$$\therefore \quad L(f * g) = \iint_{R'} e^{-s(u+v)} f(u)\,g(v)\,|J|\,du\,dv$$

$$= \int_0^\infty \int_0^\infty e^{-su}\,e^{-sv} f(u)\,g(v)\,du\,dv \qquad (\because\ |J| = |1| = 1)$$

$$= \left(\int_0^\infty e^{-su} f(u)\,du\right)\left(\int_0^\infty e^{-sv} g(v)\,dv\right)$$

$$= L(f)\,L(g).$$

$\therefore$ **L(f ∗ g) = L(f) L(g).**

Corollary 1. Let $L(f) = F(s)$ and $L(g) = G(s)$.

$\therefore$ The result of the convolution theorem can be written as

$$L(f * g) = F(s)\,G(s).$$

$$\Rightarrow \quad L^{-1}(F(s)\,G(s)) = f * g$$

$$\Rightarrow \quad L^{-1}(F(s)\,G(s)) = \int_0^t f(T)\,g(t-T)\,dT.$$

$\therefore$ **If** $\quad \mathbf{L^{-1}(F(s)) = f(t)}$ **and** $\mathbf{L^{-1}(G(s)) = g(t)}$,

then $\quad \mathbf{L^{-1}(F(s)\,G(s)) = \int_0^t f(T)\,g(t-T)\,dT.}$

This formula is used to find the inverse Laplace transform of the product of two functions.

Corollary 2. Since $f * g = g * f$, we can also write $L^{-1}(F(s)\,G(s)) = \int_0^t g(T)\,f(t-T)\,dT.$

ILLUSTRATIVE EXAMPLES

Example 1. *Find the inverse Laplace transform of the function* $\dfrac{1}{(s+3)(s-2)}$.

Sol.
$$\frac{1}{(s+3)(s-2)} = \left(\frac{1}{s+3}\right)\left(\frac{1}{s-2}\right) = F(s)\,G(s),$$

where
$$F(s) = \frac{1}{s+3} \quad \text{and} \quad G(s) = \frac{1}{s-2}$$

$$L^{-1}(F(s)) = L^{-1}\left(\frac{1}{s+3}\right) = e^{-3t} = f(t), \quad \text{say}$$

and
$$L^{-1}(G(s)) = L^{-1}\left(\frac{1}{s-2}\right) = e^{2t} = g(t), \quad \text{say}.$$

By **convolution theorem**, we have

$$L^{-1}(F(s)\,G(s)) = f * g = \int_0^t f(T)\,g(t-T)\,dT.$$

$$\therefore \quad L^{-1}\left(\frac{1}{(s+3)(s-2)}\right) = \int_0^t e^{-3T}\,e^{2(t-T)}\,dT$$

$$= \int_0^t e^{2t-5T}\,dT = \left.\frac{e^{2t-5T}}{-5}\right|_0^t = -\frac{1}{5}\left[e^{-3t} - e^{2t}\right] = \frac{\mathbf{e^{2t}}}{\mathbf{5}} - \frac{\mathbf{e^{-3t}}}{\mathbf{5}}.$$

Example 2. *Use convolution theorem to evaluate* $L^{-1}\left(\dfrac{1}{s^2(s^2+a^2)}\right)$.

Sol.
$$\frac{1}{s^2(s^2+a^2)} = \left(\frac{1}{s^2}\right)\left(\frac{1}{s^2+a^2}\right) = F(s)\,G(s), \text{ where}$$

$$F(s) = \frac{1}{s^2} \quad \text{and} \quad G(s) = \frac{1}{s^2+a^2}.$$

$$L^{-1}(F(s)) = L^{-1}\left(\frac{1}{s^2}\right) = t = f(t), \text{ say}$$

and
$$L^{-1}(G(s)) = L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}L^{-1}\left(\frac{a}{s^2+a^2}\right) = \frac{1}{a}\sin at = g(t), \text{ say}$$

By **convolution theorem,** we have

$$L^{-1}(F(s)\,G(s)) = f * g = g * f = \int_0^t g(T)\,f(t-T)\,dT$$

$$\therefore \quad L^{-1}\left(\frac{1}{s^2(s^2+a^2)}\right) = \int_0^t \left(\frac{1}{a}\sin a\mathrm{T}\right)(t-\mathrm{T})\,d\mathrm{T}$$

$$= \frac{t}{a}\int_0^t \sin a\mathrm{T}\,d\mathrm{T} - \frac{1}{a}\int_0^t \mathrm{T}\sin a\mathrm{T}\,d\mathrm{T}$$

$$= \frac{t}{a}\left(-\frac{\cos a\mathrm{T}}{a}\right)\Bigg|_0^t - \frac{1}{a}\left[\mathrm{T}\left(-\frac{\cos a\mathrm{T}}{\mathrm{a}}\right)\Bigg|_0^t - \int_0^t 1.\left(-\frac{\cos a\mathrm{T}}{a}\right)d\mathrm{T}\right]$$

$$= -\frac{t}{a^2}(\cos at - 1) - \frac{1}{a}\left[-\frac{t\cos at}{a} + 0 + \frac{1}{a}.\frac{\sin a\mathrm{T}}{a}\Bigg|_0^t\right]$$

$$= \frac{t}{a^2}(1-\cos at) + \frac{t}{a^2}\cos at - \frac{1}{a^3}\sin at - 0$$

$$= \frac{t}{a^2} - \frac{1}{a^3}\sin at = \mathbf{\frac{1}{a^3}\,(at - \sin at).}$$

Example 3. *Find the inverse Laplace transform of the function* $\frac{s}{(s^2+\pi^2)^2}$.

Sol. $$\frac{s}{(s^2+\pi^2)^2} = \left(\frac{s}{s^2+\pi^2}\right)\left(\frac{1}{s^2+\pi^2}\right) = \mathrm{F}(s)\,\mathrm{G}(s),$$

where $$\mathrm{F}(s) = \frac{s}{s^2+\pi^2} \quad \text{and} \quad \mathrm{G}(s) = \frac{1}{s^2+\pi^2}.$$

$$L^{-1}(\mathrm{F}(s)) = L^{-1}\left(\frac{s}{s^2+\pi^2}\right) = \cos \pi t = f(t), \quad \text{say}$$

and $$L^{-1}(\mathrm{G}(s)) = L^{-1}\left(\frac{1}{s^2+\pi^2}\right) = \frac{1}{\pi}L^{-1}\left(\frac{\pi}{s^2+\pi^2}\right) = \frac{1}{\pi}\sin \pi t = g(t), \quad \text{say.}$$

By **convolution theorem,** we have

$$L^{-1}(\mathrm{F}(s)\,\mathrm{G}(s)) = f * g = \int_0^t f(\mathrm{T})\,g(t-T)\,d\mathrm{T}.$$

$$\therefore\ L^{-1}\left(\frac{s}{(s^2+\pi^2)^2}\right) = \int_0^t \cos \pi\mathrm{T}\,.\,\frac{1}{\pi}\sin \pi(t-\mathrm{T})\,d\mathrm{T}$$

$$= \frac{1}{2\pi}\int_0^t [2\sin \pi\,(t-\mathrm{T})\cos \pi\mathrm{T}\,d\mathrm{T}$$

$$= \frac{1}{2\pi}\int_0^t [\sin \pi t + \sin \pi(t-2\mathrm{T})]\,d\mathrm{T} = \frac{1}{2\pi}\left[\mathrm{T}\sin \pi t - \frac{\cos \pi\,(t-2\mathrm{T})}{-2\pi}\right]_0^t$$

$$= \frac{1}{2\pi}\left[t\sin \pi t + \frac{1}{2\pi}\cos \pi t - 0 - \frac{1}{2\pi}\cos \pi t\right] = \mathbf{\frac{1}{2\pi}\,t\sin \pi t.}$$

Example 4. *Find the inverse Laplace transform of the function* $\dfrac{s+3}{(s^2+6s+13)^2}$.

Sol. $$\frac{s+3}{(s^2+6s+13)^2} = \left(\frac{s+3}{s^2+6s+13}\right)\left(\frac{1}{s^2+6s+13}\right) = \text{F}(s)\,\text{G}(s),$$

where $$\text{F}(s) = \frac{s+3}{s^2+6s+13} \quad \text{and} \quad \text{G}(s) = \frac{1}{s^2+6s+13}.$$

$$\text{L}^{-1}(\text{F}(s)) = \text{L}^{-1}\left(\frac{s+3}{s^2+6s+13}\right) = \text{L}^{-1}\left(\frac{s+3}{(s+3)^2+4}\right) = e^{-3t}\,\text{L}^{-1}\left(\frac{s}{s^2+4}\right)$$
$$= e^{-3t}\cos 2t = f(t), \text{ say}$$

and $$\text{L}^{-1}(\text{G}(s)) = \text{L}^{-1}\left(\frac{1}{s^2+6s+13}\right) = \text{L}^{-1}\left(\frac{1}{(s+3)^2+4}\right) = e^{-3t}\,\text{L}^{-1}\left(\frac{1}{s^2+4}\right)$$
$$= e^{-3t}\,\frac{\sin 2t}{2} = g(t), \text{ say.}$$

By **convolution theorem,** we have

$$\text{L}^{-1}(\text{F}(s)\,\text{G}(s)) = f * g = \int_0^t f(\text{T})\,g(t-\text{T})\,dt.$$

$$\therefore \quad \text{L}^{-1}\left(\frac{s+3}{(s^2+6s+13)^2}\right) = \int_0^t e^{-3\text{T}}\cos 2\text{T}\,.\,\frac{1}{2}e^{-3(t-\text{T})}\sin 2(t-\text{T})\,d\text{T}.$$
$$= \frac{1}{2}\int_0^t e^{-3t}\cos 2\text{T}\sin 2(t-\text{T})\,d\text{T}$$
$$= \frac{e^{-3t}}{4}\int_0^t 2\sin 2(t-\text{T})\cos 2\text{T}\,d\text{T}$$
$$= \frac{e^{-3t}}{4}\int_0^t (\sin 2t + \sin(2t-4\text{T}))\,d\text{T}$$
$$= \frac{e^{-3t}}{4}\left[(\sin 2t)\,\text{T} - \frac{\cos(2t-4\text{T})}{-4}\right]\Bigg|_0^t$$
$$= \frac{e^{-3t}}{4}\left[t\sin 2t + \frac{1}{4}\cos 2t - 0 - \frac{1}{4}\cos 2t\right] = \frac{\mathbf{1}}{\mathbf{4}}\ \mathbf{te^{-3t}\sin 2t.}$$

Example 5. *Find the inverse Laplace transform of the function* $\dfrac{s}{(s^2+a^2)^3}$.

Sol. $$\frac{s}{(s^2+a^2)^3} = \left(\frac{1}{s^2+a^2}\right)\left(\frac{1}{s^2+a^2}\right)\left(\frac{s}{s^2+a^2}\right) = \text{F}(s)\,\text{F}(s)\,\text{G}(s),$$

where $\text{F}(s) = \dfrac{1}{s^2+a^2}$ and $\text{G}(s) = \dfrac{s}{s^2+a^2}$.

$$\text{L}^{-1}(\text{F}(s)) = \text{L}^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\text{L}^{-1}\left(\frac{a}{s^2+a^2}\right) = \frac{1}{a}\sin at = f(t), \text{ say}$$

and $$\text{L}^{-1}(\text{G}(s)) = \text{L}^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at = g(t), \text{ say.}$$

By **convolution theorem,** we have

$$L^{-1}(F(s)\,F(s)) = f * f = \int_0^t f(T)\, f(t-T)\, dT.$$

$$\therefore \quad L^{-1}(F(s)\,F(s)) = \int_0^t \frac{1}{a}\sin aT \cdot \frac{1}{a}\sin a(t-T)\, dT$$

$$= \frac{1}{2a^2}\int_0^t 2\sin aT \sin(at - aT)\, dT$$

$$= \frac{1}{2a^2}\int_0^t [\cos(2aT - at) - \cos at]\, dT$$

$$= \frac{1}{2a^2}\left[\frac{\sin(2aT - at)}{2a} - T(\cos at)\right]\Bigg|_0^t$$

$$= \frac{1}{2a^2}\left[\frac{\sin at}{2a} - t\cos at - \frac{\sin(-at)}{2a} + 0\right]$$

$$= \frac{1}{2a^2}\left[\frac{\sin at}{a} - t\cos at\right] = \frac{1}{2a^3}[\sin at - at\cos at] = h(t), \text{ say.}$$

Again, by **convolution theorem,** we have

$$L^{-1}(F(s)\,F(s)\,.\,G(s)) = h * g = \int_0^t h(T)\, g(t-T)\, dT.$$

$$\therefore \quad L^{-1}\left(\frac{s}{(s^2+a^2)^3}\right) = \int_0^t \frac{1}{2a^3}(\sin aT - aT\cos aT)\cos a(t-T)\, dT$$

$$= \frac{1}{2a^3}\int_0^t (\sin aT\cos(at - aT) - aT\cos aT\cos(at - aT))\, dT$$

$$= \frac{1}{4a^3}\int_0^t (2\sin aT\cos(at - aT) - aT\,.\,2\cos aT\cos(at - aT))\, dT$$

$$= \frac{1}{4a^3}\int_0^t (\sin at + \sin(2aT - at) - aT(\cos at + \cos(2aT - at)))\, dT$$

$$= \frac{1}{4a^3}\left[\sin at\int_0^t dT + \int_0^t \sin(2aT - at)\, dT - a\cos at\int_0^t T dt \right.$$

$$\left. - a\int_0^t T\cos(2aT - at)\, dT\right]$$

$$= \frac{1}{4a^3}\left[(\sin at)T\Big|_0^t - \frac{\cos(2aT - at)}{2a}\Bigg|_0^t - (a\cos at)\frac{T^2}{2}\Bigg|_0^t \right.$$

$$\left. - a\left(T\,.\,\frac{\sin(2aT - at)}{2a}\Bigg|_0^t - \int_0^t 1\,.\,\frac{\sin(2aT - at)}{2a}\, dT\right)\right]$$

$$= \frac{1}{4a^3}\left[t \sin at - 0 - \frac{1}{2a}(\cos at - \cos at) - \frac{at^2 \cos at}{2} - \frac{1}{2}(t \sin at - 0) + \frac{1}{2}\left(\frac{-\cos(2a\mathrm{T} - at)}{2a}\right)\Bigg|_0^t\right]$$

$$= \frac{1}{4a^3}\left[t \sin at - \frac{1}{2}at^2 \cos at - \frac{1}{2}t \sin at - \frac{1}{4a}(\cos at - \cos at)\right]$$

$$= \frac{1}{4a^3}\left[\frac{1}{2}t \sin at - \frac{1}{2}at^2 \cos at\right]. = \mathbf{\frac{t}{8a^3}\,[sin\ at - at\ cos\ at].}$$

WORKING STEPS FOR SOLVING PROBLEMS

Step I. Write the given function as the product of two functions say F(s) and G(s).

Step II. Find the inverse Laplace transforms of the functions F(s) and G(s) and call these functions as $f(t)$ and $g(t)$ respectively.

Step III. Write the result of convolution theorem as

$$\mathrm{L}^{-1}(\mathrm{F}(s)\,\mathrm{G}(s)) = \int_0^t f(\mathrm{T})\, g(t - T)\, d\mathrm{T}.$$

Step IV. Substitute the values of $f(\mathrm{T})$ and $g(t - \mathrm{T})$ and simplify the integral on the right side.

TEST YOUR KNOWLEDGE

Find the inverse Laplace transform of the following functions by using convolution theorem :

1. (*i*) $\frac{1}{(s-2)(s+4)}$ (*ii*) $\frac{4}{(s-3)(s+7)}$

2. (*i*) $\frac{1}{s^2(s-1)}$ (*ii*) $\frac{1}{s(s^2+1)}$

3. (*i*) $\frac{1}{s^2(s+1)^2}$ (*ii*) $\frac{1}{s^3(s^2+1)}$

4. (*i*) $\frac{1}{(s+1)(s^2+1)}$ (*ii*) $\frac{1}{(s^2+1)^2}$

5. (*i*) $\frac{1}{s^2(s^2-a^2)}$ (*ii*) $\frac{a^2}{s(s+a)^2}$

6. (*i*) $\frac{1}{s(s+1)^3}$ (*ii*) $\frac{4}{s^3+s^2+s+1}$

7. (*i*) $\frac{s^2}{(s^2+4)^2}$ (*ii*) $\frac{s^2}{(s^2-9)^2}$

8. (*i*) $\frac{1}{s^3-a^3}$ (*ii*) $\frac{s+2}{(s^2+4s+5)^2}$

9. (*i*) $\frac{s}{(s^2+2s)^2}$ (*ii*) $\frac{1}{s(s+2)^3}$.