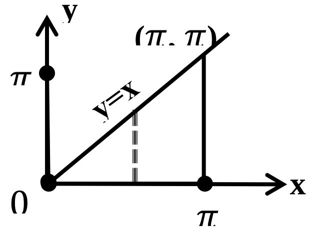
Multiple Integrals
التكاملات المضاعفة

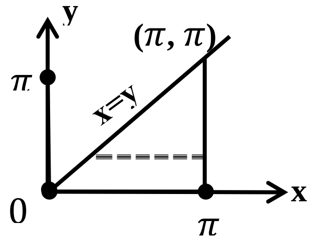
Double Integrals and Reverse the order of integration
(التكاملات الثنائية وقلب ترتيب التكامل)

Example 1: Reverse the order of integration

$$\int_0^{\pi}\int_0^{x} dy\,dx$$



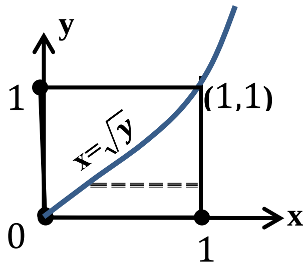
Solution: $x = 0,\ x = \pi$
and $y = 0,\ y = x$



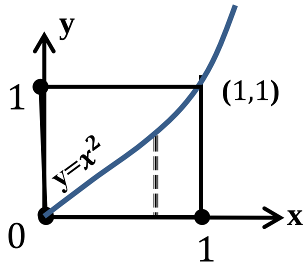
$$\therefore \int_0^{\pi}\int_0^{x} dy\,dx = \int_0^{\pi}\int_y^{\pi} dx\,dy$$

Example 2: Reverse the order of integration

$$\int_0^{1}\int_{\sqrt{y}}^{1} dx\,dy$$



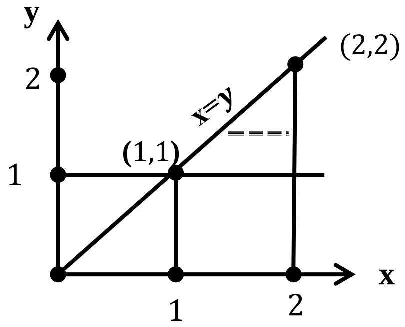
Solution: $y = 0,\ y = 1$
and $x = \sqrt{y},\ x = 1$



$$\therefore \int_0^{1}\int_{\sqrt{y}}^{1} dx\,dy = \int_0^{1}\int_0^{x^2} dy\,dx$$

Example 3: Reverse the order of integration

$$\int_1^{2}\int_1^{x} dy\,dx$$

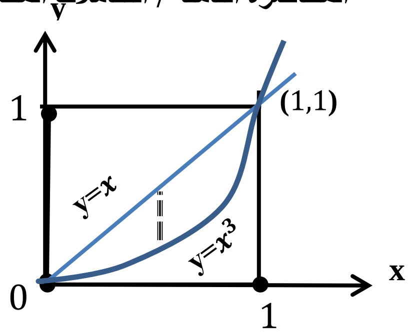


Solution: $x = 1,\ x = 2$
and $y = 1,\ y = x$

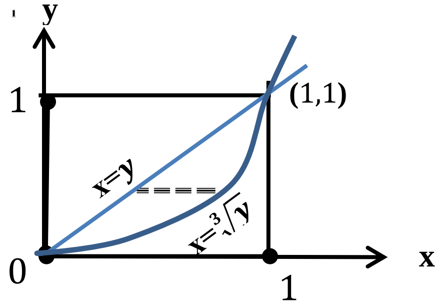
$$\therefore \int_1^{2}\int_1^{x} dy\,dx = \int_1^{2}\int_y^{2} dx\,dy$$

Example 4: Reverse the order of integration

$$\int_{0}^{1}\int_{x^3}^{x} dydx$$



Solution: $x = 0, \; 1 = 1$
and $y = x^3, \; y = x$

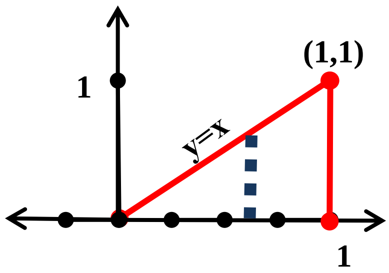


$$\therefore \int_{0}^{1}\int_{x^3}^{x} dydx = \int_{0}^{1}\int_{y}^{\sqrt[3]{y}} dxdy$$

Example 1: Evaluate (قدر او خمن) the double integral, then reverse the order of integration (اقلب حدود التكامل) and evaluate it.

$\int_0^1 \int_0^x xy\, dydx$

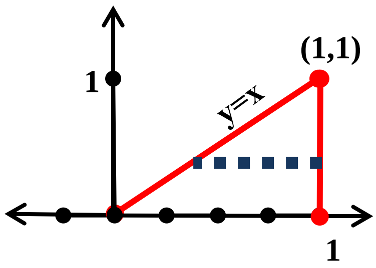
Solution: $x = 0, \; x = 1$ and $y = 0, \; y = x$



$$\int_0^1 \int_0^x xy\, dydx = \int_0^1 \frac{xy^2}{2}\Big|_0^x dx = \int_0^1 \frac{x^3}{2} dx = \frac{x^4}{8}\Big|_0^1 = \frac{1}{8}$$

Reverse the order of integration

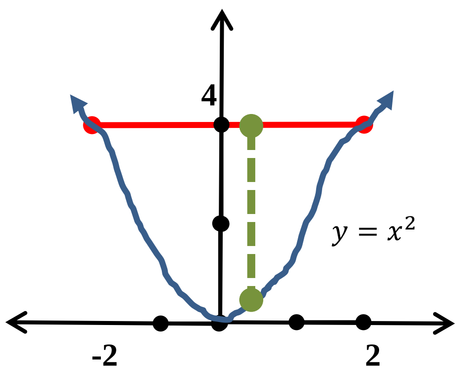
$$\int_0^1 \int_y^1 xy\, dxdy = \int_0^1 \frac{yx^2}{2}\Big|_y^1 dy = \int_0^1 \left[\frac{y}{2} - \frac{y^3}{2}\right] dy$$



$$\frac{y^2}{4} - \frac{y^4}{8}\Big|_0^1 = \frac{1}{4} - \frac{1}{8} = \frac{2-1}{8} = \frac{1}{8}$$

Example 2: Evaluate the double integral, then reverse the order of integration and evaluate it.

$$\int_{-2}^{2}\int_{x^2}^{4} x\, dydx$$



Solution: $x = -2, \; x = 2$ and $y = x^2, \; y = 4$

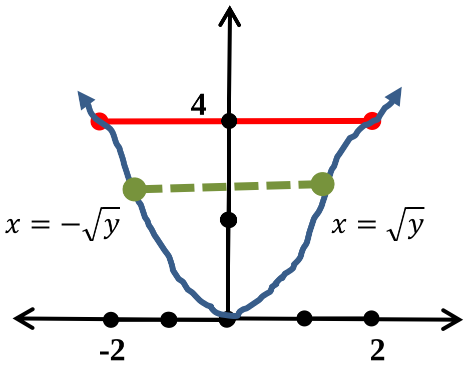
$$\int_{-2}^{2}\int_{x^2}^{4} x\, dydx = \int_{-2}^{2} xy|_{x^2}^{4}\, dx = \int_{-2}^{2} x[4 - x^2]\, dx$$

$$= \int_{-2}^{2} [4x - x^3] dx = 2x^2 - \frac{x^4}{4}\Big|_{-2}^{2} = \mathbf{8 - 4 - [8 - 4] = 0}$$

Reverse the order of integration

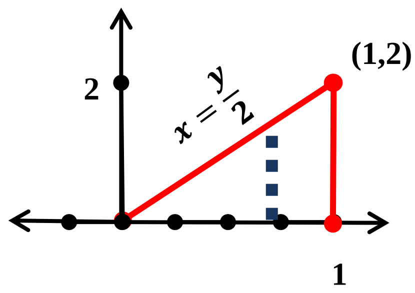
$$\int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} x\,dxdy = \int_0^4 \frac{x^2}{2}\Big|_{-\sqrt{y}}^{\sqrt{y}} dy = \int_{-2}^{2} \left[\frac{y}{2} - \frac{y}{2}\right] dy$$

$$= \int_0^4 0\ dy = 0$$



Example 3: Evaluate the double integral

$\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2}\, dxdy$



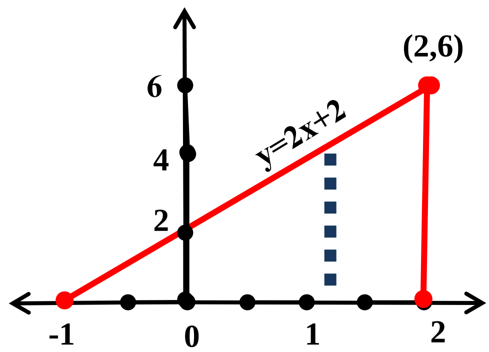
Solution: $y = 0,\ y = 2$ and $x = \frac{y}{2},\ x = 1$

$\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2}\, dxdy = \int_0^1 \int_0^{2x} e^{x^2}\, dydx$

$$\int_0^1 \int_0^{2x} e^{x^2}\, dydx = \int_0^1 e^{x^2}\, y\big|_0^{2x} dx = \int_0^1 2xe^{x^2} dx = e^{x^2}\big|_0^1 = e - 1$$

Example 4: Evaluate the double integral and reverse the order of integration and evaluate it

$$\int_{-1}^{2} \int_0^{2x+2} dy\,dx$$



Solution: $x = -1,\ x = 2$ and $y = 2x + 2$

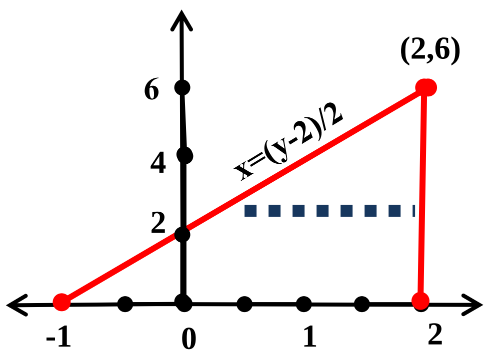
$$\int_{-1}^{2} \int_0^{2x+2} dy\,dx = \int_{-1}^{2} y\big|_0^{2x+2} dx = \int_{-1}^{2} (2x+2)dx =$$

$$= x^2 + 2x\big|_{-1}^{2} = 4 + 4 - [1 - 2] = 9$$

Reverse the order of integration

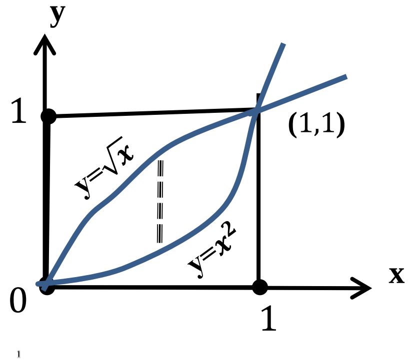
$$\int_0^6 \int_{\frac{y-2}{2}}^{2} dx\,dy = \int_0^6 x\big|_{\frac{y-2}{2}}^{2} dy = \int_0^6 \left(2 - \frac{y-2}{2}\right) dy =$$

$$2y - \left(\frac{y^2}{4} - y\right)\Big|_0^6 = 12 - (9 - 6) = 9$$



Example 5: Evaluate the double integral and reverse the order of integration and evaluate it

$$\int_0^1 \int_{x^2}^{\sqrt{x}} dy\, dx$$

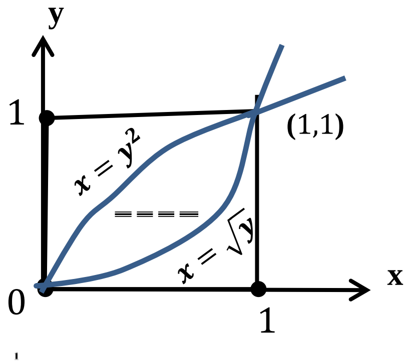


Solution: $x = 0,\ x = 1$ and $y = x^2,\ y = \sqrt{x}$

$$\int_0^1 \int_{x^2}^{\sqrt{x}} dy\, dx = \int_0^1 y|_{x^2}^{\sqrt{x}} dx = \int_0^1 (\sqrt{x} - x^2) dx =$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3}\Bigg|_0^1 = \frac{1}{\frac{3}{2}} - \frac{1}{3} = \frac{1}{3}$$

Reverse the order of integration

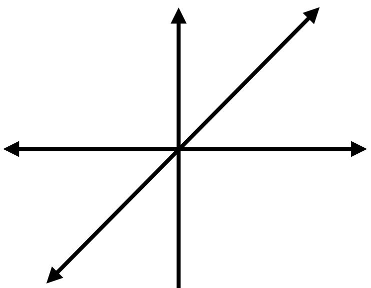


$$\int_0^1 \int_{y^2}^{\sqrt{y}} dx\, dy = \int_0^1 x|_{y^2}^{\sqrt{y}} dy = \int_0^1 (\sqrt{y} - y^2) dy =$$

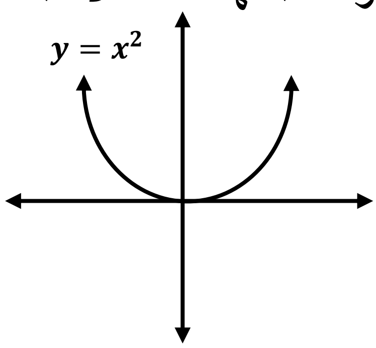
$$\frac{y^{\frac{3}{2}}}{\frac{3}{2}} - \frac{y^3}{3}\Bigg|_0^1 = \frac{1}{\frac{3}{2}} - \frac{1}{3} = \frac{1}{3}$$

بعض الرسومات المهمة للتكاملات المضاعفة

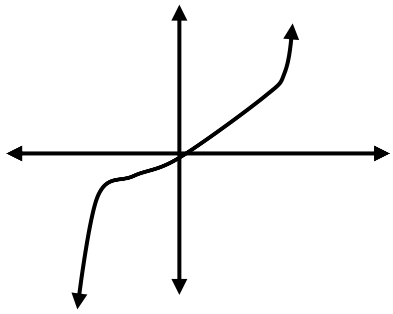
$y = x$



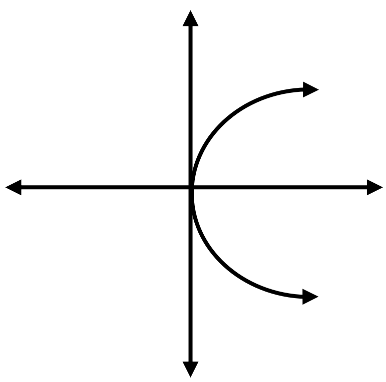
$y = x^2$



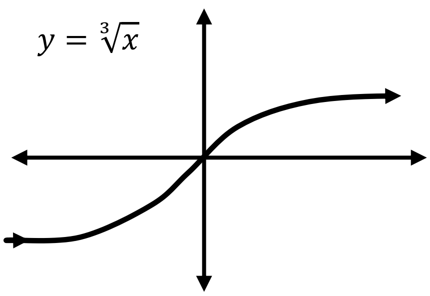
$y = x^3$



$y = \pm\sqrt{x}$



$y = \sqrt[3]{x}$



$y = e^x$ and $y = \ln x$