Mean Value Theorem

Def.:

Let f be differentable on (a, b), and Continuous on [a, b], Then there is at least one number c in (a, b) such that :-

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Ex

Does the following functions satisfies the mean value theorem :-

1. $f(x) = 3 - 3x^2$; [0, 1]

Solu ① f(x) is continuous on [0, 1], since it is polynomial

② $f'(x) = -6x$; exist on [0, 1]

∴ f(x) is satisfy the mean value theorem, so there exist the number $c \in (0, 1)$

Such that :

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$-6c = \frac{0 - 3}{1 - 0}$$

$$-6c = -3 \Rightarrow c = \frac{1}{2}$$

$c \in (0, 1)$; $0 < \frac{1}{2} < 1$

2. $f(x) = \frac{1}{4}x^3 + 1 \quad ; [0,2]$

① $f(x)$ is continuous on $[0,2]$; since it is polynomial

② $f'(x) = \frac{3}{4}x^2 \quad$; exist on $(0,2)$

$\therefore f(x)$ is satisfies the mean value theorem, so there exists $c \in (0,2)$

such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\frac{3}{4}x^2 = \frac{3-1}{2-0}$$

$$\frac{3}{4}x^2 = \frac{2}{2} \Rightarrow x^2 = \frac{2}{2} \cdot \frac{4}{3} \Rightarrow x^2 = \frac{4}{3}$$

$$\therefore c^2 = \frac{4}{3} \Rightarrow c = \mp 1.33 \in (0,2)$$

$$c_1 = 1.33 \in (0,2) \; ; \; c_2 = -1.33 \notin (0,2)$$

3. $f(x) = x^{1/5} \quad , [-1,2]$

① $f(x)$ is continous on $[-1,2]$; since it is polynomial.

② $f'(x) = \frac{1}{5}x^{-\frac{4}{5}} \quad$; exist on $(-1,2)$

$\therefore f(x)$ is satisfies the mean value theorem, so there exists $c \in (-1,2)$

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad ; f(b) = (-1)^{-\frac{1}{5}} \quad \text{لا يجوز}$$

$$\frac{1}{5}x^{-\frac{4}{5}} =$$

$\therefore f(x)$ is not differentiable on $(-1,2)$

H.W.

1. Does the function $f(x) = x^{\frac{1}{5}}$ satisfies Mean Value theorem on $[\frac{1}{10}, 2]$? If so find c ?

2. Does the function $f(x) = \frac{1}{3}x^3 - 3x$ satisfies Mean Value theorem on $[-3, 3]$? If so find c?

4. $f(x) = x^3 - 3x^2 + 4$; $[0, 3]$

Sol/

① $f(x)$ is continuous on $[0,3]$, Since it is Polynomil.

② $f'(x) = 3x^2 - 6x$; exist on $(-1, 2)$

$\therefore$ $f(x)$ is satisfies the mean value theorem , So there exist $c \in (-1, 2)$

$$\therefore 3c^2 - 6c = \frac{4 - 4}{3 - 0}$$

$$\therefore 3c^2 - 6c = 0$$

$$3c(c - 2) = 0$$

$3c = 0 \Rightarrow c = 0 \in [0, 3]$

or $c - 2 = 0 \Rightarrow c = 2 \in [0, 3]$

5. $f(x) = x^3 - 2x^2 - x + 2$ on $[1,2]$

① $f(x)$ is continuous on $[1,2]$, sinc it is polynomial.

② $f'(x) = 3x^2 - 4x - 1$ exist in $(1,2)$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$3x^2 - 4x - 1 = \frac{0 - 0}{2 - 1}$$

$$\therefore 3c^2 - 4c - 1 = 0$$

$$3c^2 - 4c = 1$$

$$c(3c - 4) = 1$$

$$c = 1$$

or $3c - 4 = 1 \Rightarrow 3c = 5 \Rightarrow c = \frac{5}{3} = 1.66 \in (1,2)$

$\therefore f(x)$ is satisfies the mean value theorem

H.w.

1. $x^3 - 16x + 5$ at $[-4,4]$

2. x^3 at $[-2,2]$

3. $x^{\frac{2}{3}}$ at $[-8,8]$

4. $f(x) = \dfrac{x^2 - 4x + 3}{x - 2}$ at $[1,3]$

ملاحظة: اذا كانت الدالة كسرية وكان المقام عند اي نقطة من هذه الفترة يساوي صفر فهي غير مستمرة

$\dfrac{x^2 - 2x}{0}$; $[0,4] \Rightarrow$ is not continuous.

Partial Derivatives

Derivatives are divided into two types:-

1. Ordinary derivative

هي دالة ذات متغير واحد

$y = f(x)$

$y' = \frac{dy}{dx}$ or $\frac{df}{dx}$

2. Partial derivative

It is a function that has more than one Variable

هي دالة لها أكثر من متغير واحد

$z = f(x,y)$

$z = f(x,y,t)$

$z = f(x_1, x_2, x_3, \ldots, x_n)$

Def.:

Let it be $f(x,y)$ is a function defined in the region $A \subseteq R^2$, Then the partial derivatives are defined as follows:-

$$f_x = \frac{\partial f}{\partial x} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$f_y = \frac{\partial f}{\partial y} = f'(y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

and:-

- $$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}=f_{xx}=\lim_{\Delta x\to 0}\frac{f_x(x+\Delta x,y)-f_x(x,y)}{\Delta x}$$

- $$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}=f_{yy}=\lim_{\Delta y\to 0}\frac{f_y(x,y+\Delta y)-f_y(x,y)}{\Delta y}$$

- $$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\,\partial x}=f_{xy}=\lim_{\Delta y\to 0}\frac{f_x(x,y+\Delta y)-f_x(x,y)}{\Delta y}$$

- $$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\,\partial y}=f_{yx}=\lim_{\Delta x\to 0}\frac{f_y(x+\Delta x,y)-f_y(x,y)}{\Delta x}$$

Ex1. if the $f(x,y)=xy^2+x^2y+3$

Find f_x, f_y using definition.

Sol

$$\frac{\partial f}{\partial x}=f_x=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$$

$$\begin{aligned} f(x+\Delta x,y) &= (x+\Delta x)y^2+(x+\Delta x)^2y+3 \\ &= xy^2+\Delta xy^2+(x^2+2x\Delta x+(\Delta x)^2)y+3 \\ &= xy^2+y^2\Delta x+x^2y+2xy\Delta x+(\Delta x)^2+3 \end{aligned}$$

$f(x+\Delta x,y)-f(x,y)=$

$=xy^2+y^2\Delta x+\cancel{x^2y}+2xy\Delta x+(\Delta x)^2\cancel{+3}-xy^2-\cancel{x^2y}\cancel{-3}$

$=y^2\Delta x+2xy\Delta x+(\Delta x)^2$

$=\Delta x(y^2+2xy+\Delta x)$

$$\frac{\partial f}{\partial x}=f_x=\lim_{\Delta x\to 0}\frac{\Delta x(y^2+2xy+\Delta x)}{\Delta x}=y^2+2xy$$

$$\frac{\partial f}{\partial y} = f_y = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

$$f(x, y+\Delta y) = x(y+\Delta y)^2 + x^2(y+\Delta y) + 3$$
$$= x(y^2 + 2y\Delta y + (\Delta y)^2) + x^2 y + x^2 \Delta y + 3$$
$$= xy^2 + 2xy\Delta y + x(\Delta y)^2 + x^2 y + x^2 \Delta y + 3$$

$$f(x, y+\Delta y) - f(x, y)$$

$$= \cancel{xy^2} + 2xy\Delta y + x(\Delta y)^2 + \cancel{x^2 y} + x^2\Delta y \cancel{+3} \cancel{-xy^2} \cancel{-x^2 y} \cancel{-3}$$

$$= 2xy\Delta y + x(\Delta y)^2 + x^2 \Delta y$$
$$= \Delta y(2xy + x\Delta y + x^2)$$

$$\frac{\partial f}{\partial y} = f_y = \lim_{\Delta y \to 0} \frac{\Delta y(2xy + x\Delta y + x^2)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} 2xy + x\Delta y + x^2$$

$$= 2xy + x^2$$

2. If $f(x, y) = xy\, e^{x^2+y^2}$

Find f_y ; f_x

Solu

$$f_x = xy \cdot e^{x^2+y^2}\, 2x + e^{x^2+y^2} \cdot y = y\, e^{x^2+y^2}(2x^2+1)$$

$$f_y = xy \cdot e^{x^2+y^2}(2y) + e^{x^2+y^2}(x) = x\, e^{x^2+y^2}(2y^2+1)$$

3. If $f(x,y) = e^{\sin(y/x)}$, Find f_y , f_x when $x \neq 0$

Solu

$$f_x = e^{\sin(y/x)} \cdot \cos(y/x) \cdot \left(\frac{-y}{x^2}\right)$$

$$= \frac{-y}{x^2} \cdot \cos(y/x) \cdot e^{\sin(y/x)}$$

$$f_y = e^{\sin(y/x)} \cdot \cos(y/x) \cdot \left(\frac{1}{x}\right)$$

$$= \frac{1}{x} \cdot \cos(y/x)\, e^{\sin(y/x)}$$

4. If $f(x,y) = xy^4 - 2x^2y^5 + 4x^6 - 3y^2$

Find $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ and $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$; f_{xy} , f_{yx}

Solu

$$\frac{\partial f}{\partial x} = y^4 - 4xy^5 + 24x^5 = f_x$$

$$\frac{\partial f}{\partial y} = 4xy^3 - 10x^2y^4 - 6y = f_y$$

$$\frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 4y^3 - 20xy^4 = 4y^3(1-5xy) = f_{yx}$$

$$\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 4y^3 - 20xy^4 = 4y^3(1-5xy) = f_{xx}$$