8th Lecture

Theorem (2.2): If (X, τ) is a topological space and $X^* \subset X$. Then (X^*, τ^*) is a topology.

Proof:

(1) Since $\emptyset \in \tau$ and $\emptyset \cap X^* = \emptyset \Rightarrow \emptyset \in \tau^*$

Also since $X \in \tau$ and $X \cap X^* = X^* \Rightarrow X^* \in \tau^*$

(2) Let $G_\lambda^* \in \tau^*, \ \forall \lambda \in \Lambda$

$\Rightarrow G_\lambda^* = G_\lambda \cap X^*, \forall \ \lambda \in \Lambda$

Now

$$\Rightarrow \bigcup_{\lambda \in \Lambda} G_\lambda^* = \bigcup_{\lambda \in \Lambda} (G_\lambda \cap X^*)$$

$$= (\bigcup_{\lambda \in \Lambda} G_\lambda) \cap (\bigcup X^*)$$

$$= (\bigcup_{\lambda \in \Lambda} G_\lambda) \cap X^* \in \tau^* \text{ because } \bigcup_{\lambda \in \Lambda} G_\lambda \in \tau$$

$$\Rightarrow \bigcup_{\lambda \in \Lambda} G_\lambda^* \in \tau^*$$

(3) $G_i^* \in \tau, \quad 1 \le i \le n$

$\Rightarrow G_i^* = G_i \cap X^*, \quad 1 \le i \le n$

$\Rightarrow \bigcap_{i=1}^n G_i^* = \bigcap_{i=1}^n (G_i \cap X^*)$

$\Rightarrow \bigcap_{i=1}^n G_i^* = \bigcap_{i=1}^n G_i \cap X^*$

But $\bigcap_{i=1}^n G_i \in \tau$

$\Rightarrow \bigcap_{i=1}^n G_i \cap X^* \in \tau^*$

$\Rightarrow \bigcap_{i=1}^n G_i \in \tau^*$

Hence (X^*, τ^*) is a topology on X^*.

Theorem (2.3): If (X^*, τ^*) is a topological subspace of (X, τ) and if $A \subset X^* \subset X$ is τ open (i.e. A is open w.r.t. τ) Hence A is τ^*-open.

Proof:

We have $\tau^* = \{G^* : G^* = G \cap X^*, G \in \tau\}$

We are given $A \subset X^*$, $A \in \tau$

$\Rightarrow A \cap X^* = A$

We have $A \cap X^* \in \tau^* \Rightarrow A \in \tau^*$

$\Rightarrow A$ is τ^*- open

Note (2.1): The converse of the above theorem is not true is general as shown in the following example.

Example (2.5): Let $X = \{a, b, c, d, e\}$ and

$\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{b, c, d\}, \{a, c, d\}, \{a, b, c, d\}, \{b, c, d, e\}, X\}$

Let $X^* = \{a, d, e\}$ and $\tau^* = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{d, e\}, X^*\}$

Solution:

$\{d\} \subset X^*$ and $\{d\}$ is τ^* - open

But $\{d\}$ is not τ - open

Theorem (2.4): If E is a subset of (X^*, τ^*) and (X^*, τ^*) is a topological subspace of (X, τ). Then

$$\overline{E^*} = X^* \cap \bar{E}$$

Proof: We have

$R.H.S. = X^* \cap \bar{E}$

$= X^* \cap \{\bigcap F : F \in \Psi\}$, where Ψ is the family of closed $F \supset E$

$= \bigcap \{X^* \cap F : F \in \Psi\}$

$= \bigcap\{F^* : F^* \in \Psi^*\}$, where Ψ^* is the family of closed $F^* \supset E^*$

$= \bigcap_{\forall F^* \supset E^*} F^*$, where F^* is closed

$= E^* = L.H.S.$

Theorem (2.5): If (X^{**}, τ^{**}) is a topological subspace of and (X^*, τ^*) and (X^*, τ^*) is a topological subspace of (X, τ). Then (X^{**}, τ^{**}) is a topological subspace of (X, τ).

Proof: We have

$\tau^{**} = \{G^{**} : G^{**} = X^{**} \cap G^* \,; G^* \in \tau^*\}$

$\quad = \{G^{**} : G^{**} = X^{**} \cap (X^* \cap G) \,; G \in \tau\}$

$\quad = \{G^{**} : G^{**} = (X^{**} \cap X^*) \cap G \,; G \in \tau\}$

$\tau^{**} = \{G^{**} : G^{**} = X^{**} \cap G \,; G \in \tau\}$

$\therefore$ (X^{**}, τ^{**}) is a topological subspace of (X, τ).

Exercises (2.1): (Homework)

(1) If (X^*, τ^*) is a topological subspace of (X, τ) and if $A \subset X^* \subset X$ is τ-closed (i.e. A is closed w.r.t. τ) Then A is τ^*-closed.

(2) Give an example to show that the converse of (1) is not true in general.
