(12) $T(\alpha x) = A(\alpha x)$
$= \alpha A x$
$= \alpha T(x)$

So T is linear operator.

now

$$\|T(x)\|^2 = \|y\|^2 = \sum_{i=1}^{r} y_i^2$$

$$= \sum_{i=1}^{r} \left(\sum_{j=1}^{n} a_{ij} x_j \right)^2$$

by using Cauchy Schwarz inequality

$$\leq \sum_{i=1}^{r} \left[\left(\sum_{j=1}^{n} a_{ij}^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} x_j^2 \right)^{\frac{1}{2}} \right]^2$$

$$= \sum_{i=1}^{r} \left[\left(\sum_{j=1}^{n} a_{ij}^2 \right)^{\frac{1}{2}} \|x\| \right]^2$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{n} a_{ij}^2 \|x\|^2$$

$$= \|x\|^2 \sum_{i=1}^{r} \sum_{j=1}^{n} a_{ij}^2$$

noting that the double sumintion does not depend on ~~x~~ x, we can write our result in the form

$\|T(x)\|^2 \leq c^2 \|x\|^2$ with $c^2 = \sum_{i=1}^{r} \sum_{j=1}^{n} a_{ij}^2$

This gives ~~a~~ completes the proof that T is bounded.

Lemma (linear combinations)

Let $\{x_1, x_2, \cdots, x_n\}$ be a linearly independent set of vectors in a normed space X (of any dimension). Then there is a number $c>0$ such that for every choice of scalars $\alpha_1, \cdots, \alpha_n$ we have:

$$\| \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \| \geq c(|\alpha_1| + \cdots |\alpha_n|)$$

$$\| \sum_{i=1}^{n} \alpha_i x_i \| \geq c \left(\sum_{i=1} |\alpha_i| \right) \qquad c>0$$

Theorem (finite dimension)

If a normed space X is finite dimensional. Then every linear operator on X is bounded.

proof: Let $\dim(X) = n$ and $\{e_1, \cdots, e_n\}$ is a basis for X and T is a linear operator on X. for $x \in X$ we can write x

$$x = \sum_{i=1}^{n} \alpha_i e_i$$

(H) $\|T(x)\| = \|T(\sum_{i=1}^{n} \alpha_i e_i)\|$

$= \|\sum_{i=1}^{n} \alpha_i T(e_i)\|$ since T is linear

$\leq \sum_{i=1}^{n} \|\alpha_i T(e_i)\|$ by triangular inequality

Since $\|T(e_i)\| \leq \max_{1\leq i\leq n}\{\|T(e_i)\|\}$ for all $1 \leq i \leq n$

set $c_1 = \max_{1\leq i\leq n}\{\|T(e_i)\|\}$

so $\|T(e_i)\| \leq c_1$ for all $1 \leq i \leq n$ —— ①

$\|T(x)\| \leq \sum_{i=1}^{n} \|\alpha_i T(e_i)\|$

$= \sum_{i=1}^{n} |\alpha_i| \, \|T(e_i)\|$

by using lemma * $\exists\, c_2 > 0$ such that

$$\sum_{i=1}^{n} |\alpha_i| c_2 \leq \|\sum_{i=1}^{n} \alpha_i e_i\| = \|x\|$$

so $\sum_{i=1}^{n} |\alpha_i| \leq \frac{1}{c_2}\|x\|$ —— ②

~~Lemma (Norm). Let T be bounded~~

$$\|T(x)\| \leq \sum_{i=1}^{n} |\alpha_i| \, \|T(e_i)\|$$

$$\leq \sum_{i=1}^{n} |\alpha_i| \, c_1 \quad \text{from } ①$$

$$= c_1 \sum_{i=1}^{n} |\alpha_i|$$

$$\leq \frac{c_1}{c_2} \|x\| \quad \text{from } ②$$

set $c = \dfrac{c_1}{c_2}$ we have

$$\|T(x)\| \leq c\|x\| \quad \forall x \in X$$

Therefore T is bounded.

Example Let $T: R^2 \longrightarrow R^3$

$T(x_1, x_2) = (x_2, x_1, x_1)$ $\quad \|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}$, $\|(x_1, x_2, x_3)\| = \max\{|x_1|, |x_2|, |x_3|\}$

is T is linear or bounded operator.

Solution Let $x, y \in R^2$, then

$x = (x_1, x_2)$

$y = (y_1, y_2)$

$x + y = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$

(16) $T(x+y) = T(x_1+y_1, x_2+y_2) = (x_2+y_2, x_1+y_1, x_1+y_1)$

$= (x_2, x_1, x_1) + (y_2, y_1, y_1)$

$= T(x_1, x_2) + T(y_1, y_2)$

$= T(x) + T(y)$

Let $\alpha \in R$,

$T(\alpha x) = T(\alpha(x_1, x_2)) = T(\alpha x_1, \alpha x_2)$

$= (\alpha x_2, \alpha x_1, \alpha x_1)$

$= \alpha(x_2, x_1, x_1)$

$= \alpha T(x_1, x_2)$

$= \alpha T(x)$

T is linear operator. By theorem is bounded we will explain it is bounded

$\|T(x)\| = \|T(x_1, x_2)\|$

$= \|(x_2, x_1, x_1)\|$

$= \max\{|x_2|, |x_1|, |x_1|\}$

$= \max\{|x_1|, |x_2|\}$

$= \|(x_1, x_2)\|$

$= \|x\|$

$\therefore \|T(x)\| = \|x\| \quad \forall x \in X$

T is bounded with $c = 1$.

In ~~chapter~~ section 2.1 linear operator we show that $(L(X,Y),+,\cdot,K)$ ~~is~~ the ~~...~~ vector space of all linear operator from vector space X into vector space Y.

In this secation we define norm for operator so when X and Y are normed ~~...~~ space we can find find norm for any bounded operator from X into Y.

now we will take a sub set from $L(X,Y)$ which all operator which be bounded and denoted it by $BL(X,Y)$, so

$$BL(X,Y) \subseteq L(X,Y).$$

In general $L(X,Y)$ is not normed space because there is some operator it is linear but not bounded. But $BL(X,Y)$ the set of all linear bounded operators it will be normed space with operator norm. ~~and~~

Theorem

$(BL(X,Y), \|\cdot\|)$ is normed space with operator norm where X, Y are normed spaces.

proof,

If first we must prove that $BL(X,Y)$ is vector space, to show that we prove that $BL(X,Y)$ is a subspace of $L(X,Y)$.

Let $T, S \in BL(X,Y)$, to prove that $T+S \in BL(X,Y)$ we must show that

① $T+S$ is linear operator.
② $T+S$ is bounded operator.

To prove ①, since $BL(X,Y) \subseteq L(X,Y)$
so $S, T \in L(X,Y)$ and $L(X,Y)$ is vector space
so $S+T \in L(X,Y)$. Then $S+T$ is linear operator

To prove ②, since $S, T \in BL(X,Y)$, so S, T is bounded operator, that is mean

$\|T(x)\| \leq \|T\| \|x\| \quad \forall x \in X$

$\|S(x)\| \leq \|S\| \|x\|$

$$\|(T+S)(x)\| = \|T(x)+S(x)\|$$
$$\le \|T(x)\| + \|S(x)\|$$

by traingular inquality

$$\le \|T\|\,\|x\| + \|S\|\,\|x\|$$

since T, S are bounded operator

$$= (\|T\|+\|S\|)\,\|x\|$$

where $c_1 = \|T\|+\|S\|$

$\therefore$ $T+S$ is bounded operator $\quad \forall x \in X$

so $T+S \in BL(X,Y)$.

now let $\alpha \in K$, we to prove that $\alpha T \in BL(X,Y)$

$\because$ $BL(X,Y) \le L(X,Y)$ implies that $T \in L(X,Y)$, since $L(X,Y)$ is vector space

so $\alpha T \in L(X,Y)$, this mean αT is linear operator.

now $\|(\alpha T)(x)\| = \|\alpha T(x)\|$
$$= |\alpha|\,\|T(x)\|$$
$$\le |\alpha|\,\|T\|\,\|x\|$$

when $c_2 = |\alpha|\,\|T\|$ $\quad \forall x \in X$

$\therefore$ αT is bounded operator

So $\alpha T \in BL(X,Y)$.

~~Then~~ Therefore $BL(X,Y)$ is a subspace of $L(X,Y)$.

(20)

To prove that $(BL(X,Y), \|\cdot\|)$ is normed with operator norm

$$\|T\| = \sup_{\substack{x \in X \\ \|x\| = 1}} \{\|T(x)\|\}$$

(1) $$\|T\| = \sup_{\substack{x \in X \\ \|x\| = 1}} \{\|T(x)\|\}$$

$\because \|T(x)\| \geq 0$ since $T(x) \in Y$ and Y is normed it true $\forall x \in X$ ~~$\forall T \in BL(X,Y)$~~

$\therefore \|T\| \geq 0$

That is for all $T \in BL(X,Y)$.

(2) $$\|T\| = 0 \longleftrightarrow \sup_{\substack{x \in X \\ \|x\| = 1}} \{\|T(x)\|\} = 0$$

$$\longleftrightarrow \|T(x)\| = 0 \quad \forall x \in X$$

$$\longleftrightarrow T = O \quad \text{zero operator}$$

(3) $$\|\alpha T\| = \sup_{\substack{x \in X \\ \|x\| \leq 1}} \{\|(\alpha T)(x)\|\}$$

$$= \sup_{\substack{x \in X \\ \|x\| = 1}} \{\|\alpha T(x)\|\} \quad \text{since } Y \text{ is normed space}$$

$$= \sup_{\substack{x \in X \\ \|x\| = 1}} \{|\alpha| \|T(x)\|\}$$

$$= |\alpha| \sup_{\substack{x \in X \\ \|x\| = 1}} \{\|T(x)\|\} = |\alpha| \|T\|$$

(4) ~~$\|(T+S)(x)\| \leq \|T(x)\|$~~

$$\|T+S\| = \sup_{\substack{x \in X \\ \|x\|=1}} \{\|(T+S)(x)\|\}$$

$$= \sup_{\substack{x \in X \\ \|x\|=1}} \{\|T(x)+S(x)\|\}$$

$$\leq \sup_{\substack{x \in X \\ \|x\|=1}} \{\|T(x)\| + \|S(x)\|\}$$

$$= \sup_{\substack{x \in X \\ \|x\|=1}} \{\|T(x)\|\} + \sup_{\substack{x \in X \\ \|x\|=1}} \{\|S(x)\|\}$$

$$= \|T\| + \|S\|$$

$$\therefore \|T+S\| \leq \|T\| + \|S\|$$

so $(BL(X,Y), \|\cdot\|)$ is normed space over the field $\mathbb{K}$.

Theorem

Let X, Y, Z be ~~two~~ normed spases, If

$$T_1 : X \longrightarrow Y$$
$$T_2 : Y \longrightarrow Z$$
$$T : X \longrightarrow X$$

are bounded operator then.

$$\|T_2 T_1\| \leq \|T_2\| \|T_1\|$$
$$\|T^n\| \leq \|T\|^n \qquad \forall n \in N$$

proof: Let $x \in X$

$$\|(T_2 T_1)(x)\| = \|T_2(T_1(x))\|$$
$$\leq \|T_2\| \|T_1(x)\| \quad \text{since } T_2 \text{ is bounded}$$
$$\leq \|T_2\| \|T_1\| \|x\| \quad \text{since } T_1 \text{ is bounded}$$
$$\forall x \in X$$

$$\|T_2 T_1\| = \sup_{\substack{x \in X \\ \|x\| \leq 1}} \{ \|(T_2 T_1)(x)\| \}$$
$$\leq \sup_{\substack{x \in X \\ \|x\| \leq 1}} \{ \|T_2\| \|T_1\| \|x\| \}$$
$$= \|T_2\| \|T_1\| \sup_{\substack{x \in X \\ \|x\| \leq 1}} \{ 1 \}$$
$$= \|T_2\| \|T_1\|$$

$$\therefore \|T_2 T_1\| \leq \|T_2\| \|T_1\|$$

when $n=1$, it is clearly that

$\|T\| \leq \|T\|$

when $n=2$

$\|T^2\| = \|TT\| \leq \|T\|\,\|T\|$ from first proof.

$= \|T\|^2$

so $\|T^2\| \leq \|T\|^2$

now let it is true when $n=k$

$\|T^k\| \leq \|T\|^k$ ——— $*$

~~now the~~ we to prove it when $n=k+1$

$$\|T^{k+1}\| = \sup_{\substack{x\in X\\ \|x\|\leq 1}} \{\|T^{k+1}(x)\|\}$$

$$= \sup_{\substack{x\in X\\ \|x\|\leq 1}} \{\|T(T^k(x))\|\}$$

$$\leq \sup_{\substack{x\in X\\ \|x\|\leq 1}} \{\|T\|\,\|T^k(x)\|\}$$

$$= \|T\| \sup_{\substack{x\in X\\ \|x\|\leq 1}} \{\|T^k(x)\|\}$$

$$\cancel{= \|T\|\,\|T^k\|}$$

$$= \|T\|\,\|T^k\|$$

$$\leq \|T\|\,\|T\|^k \quad \text{from } *$$

$$= \|T\|^{k+1}$$

so $\|T^n\| \leq \|T\|^n$ for any $n \in \mathbb{N}$

Example

Let $T_1 : R^3 \longrightarrow R$

such that $T(x) , T(x_1, x_2, x_3) = x_1 + x_2 + x_3$

$x \in R$

$T_2 : R \longrightarrow R^2$ $\|(x_1,x_2,x_3)\|_3 = \|x\|_3 = |x_1| + |x_2| + |x_3|$

such that $T(y) = (y, y)$ $\|(x_1, x_2)\|_2 = |x_1| + |x_2|$

where $x \in R^3$, $x_1, x_2, x_3, y \in R$

find $T_2 T_1$ is linear operator?

is bounded operator. ?

Solution

$T_2 T_1 : R^3 \longrightarrow R^2$

has domain R^3 and condomain R^2

$$
\begin{aligned}
T_2 T_1 (x) &= T_2 T_1 (x_1, x_2, x_3) \\
&= T_2 (T_1 (x_1, x_2, x_3)) \\
&= T_2 (x_1 + x_2 + x_3) \\
&= (x_1 + x_2 + x_3 , x_1 + x_2 + x_3)
\end{aligned}
$$

So

$T_2 T_1 (x_1, x_2, x_3) = (x_1 + x_2 + x_3 , x_1 + x_2 + x_3)$

Let $x, y \in R^3$ such that $x = (x_1, x_2, x_3)$

$y = (y_1, y_2, y_3)$

when $x_i, y_i \in R$ $1 \le i \le 3$

$$
\begin{aligned}
T_2 T_1 (x+y) &= T_2 T_1 ((x_1, x_2, x_3) + (y_1, y_2, y_3)) \\
&= T_2 T_1 (x_1 + y_1 , x_2 + y_2 , x_3 + y_3) \\
&= (x_1 + y_1 + x_2 + y_2 + x_3 + y_3 , x_1 + y_1 + x_2 + y_2 + x_3 + y_3)
\end{aligned}
$$

23 . 3

$= (x_1+x_2+x_3\ ,\ x_1+x_2+x_3) + (y_1+y_2+y_3\ ,\ y_1+y_2+y_3)$

$= T_2T_1(x) + T_2T_1(y)$

now $T_2T_1(\alpha x) = T_2T_1\,\alpha(x_1, x_2, x_3)$

$= T_2T_1(\alpha x_1, \alpha x_2, \alpha x_3)$

$= (\alpha x_1 + \alpha x_2 + \alpha x_3\ ,\ \alpha x_1 + \alpha x_2 + \alpha x_3)$

$= \alpha(x_1 + x_2 + x_3\ ,\ x_1 + x_2 + x_3)$

$= \alpha\, T_2T_1(x)$

$\therefore T_2T_1$ is linear operator

T_2T_1 is bounded why?

$\|T_2T_1(x)\|_2 = \|T_2T_1(x_1, x_2, x_3)\|_2$

$= \|(x_1+x_2+x_3\ ,\ x_1+x_2+x_3)\|_2$

$= |x_1+x_2+x_3| + |x_1+x_2+x_3|$

$\leq |x_1| + |x_2| + |x_3| + |x_1| + |x_2| + |x_3|$

$= 2(|x_1| + |x_2| + |x_3|)$

$= 2\|x\|_3$

so $\|T_2T_1(x)\|_2 \leq 2\|x\|_3$

T_2T_1 is bounded with $c = 2$.

(24)

Theorem Let X be a normed space and Y be a Banach space. Then $BL(X, Y)$ is Banach space.

proof:

Let $(T_n)_{n=1}^{\infty}$ be a Cauchy sequence in $BL(X,Y)$

that is mean, for any $\epsilon > 0$, $\exists$ $k_1 \in N$ such that

$$\| T_n - T_m \| < \epsilon \qquad \forall n, m > k_1$$

that mean $\| T_n - T_m \| = \sup_{\substack{x \in X \\ \|x\| = 1}} \| (T_n - T_m)(x) \| < \epsilon$

So $\| T_n - T_m (x) \| = \| T_n(x) - T_m(x) \| < \epsilon$

$\therefore (T_n(x))_{n=1}^{\infty}$ is a sequence in Y and it is Cauchy sequence so it is convergent, since Y is Banach space

Let $T(x)$ is the convergent point so

$$T_n(x) \longrightarrow T(x) \qquad \forall x \in X$$

that is mean $\forall$ $\epsilon > 0$, $\exists$ $k_2 \in N$ such that

$$\| T_n(x) - T(x) \| < \epsilon \qquad \forall n > k$$

$$\| T_n - T \| = \sup_{\substack{x \in X \\ \|x\| = 1}} \{ \| (T_n - T)(x) \| \}$$

$$= \sup_{\substack{x \in X \\ \|x\| = 1}} \{ \| T_n(x) - T(x) \| \} < \sup_{\substack{x \in X \\ \|x\| = 1}} \{ \epsilon \}$$

$$= \epsilon$$

$\| T_n - T \| < \epsilon \qquad \forall n > k$

$\therefore\ T_n \longrightarrow T$, so $(T_n)_{n=1}^{\infty}$ is convergent to T.

now must we to show that $T \in BL(X, Y)$

Let $\alpha, \beta \in K$, $x, y \in X$. Then

$$T(\alpha x + \beta y) = \lim_{n\to\infty} T_n(\alpha x + \beta y)$$

$$= \lim_{n\to\infty} \alpha T_n(x) + \beta T_n(y) \qquad T_n \text{ is linear operator}$$

$$= \lim_{n\to\infty} \alpha T_n(x) + \lim_{n\to\infty} \beta T_n(y)$$

$$= \alpha \lim_{n\to\infty} T_n(x) + \beta \lim_{n\to\infty} T_n(y)$$

$$= \alpha T(x) + \beta T(y)$$

So $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$

$\therefore\ T$ is linear operator.

To prove T is bounded, let $x \in X$ with

~~Since we can write $T(x) = \lim_{n\to\infty} T_n(x)$~~

$$\|T_n(x)\| \leq C\|x\| \quad \text{since } (T_n)_{n=1}^{\infty} \text{ is in } BL(X,Y)$$

by taking limit $n \to \infty$ to both the inequality

$$\|T(x)\| \leq C\|x\| \quad \forall x \in X$$

so T is bounded so $T \in BL(X,Y)$

$\therefore BL(X,Y)$ is Banach space.

Continuous operators

المؤثرات المستمرة

Let $T : D(T) \longrightarrow Y$ be any operator not necessarily linear where $D(T) \subseteq X$ and X, Y are normed space.

The operater T is called continuous operator at point $x_0 \in D(T)$. if ~~and only if~~ for every $\epsilon > 0$ there exists $\delta > 0$ such that

$\| T(x) - T(x_0) \| < \epsilon$ whenever

$\| x - x_0 \| < \delta$ for all $x \in D(T)$

That is mean if $\| x - x_0 \| < \delta$ must $\| T(x) - T(x_0) \| < \epsilon$

we called T is continuous if T is continuous at every point $x \in D(T)$

88

Example

Let $I : X \longrightarrow X$ is the identity operator

where X is any normed space

~~Let $\|x - x_0\| < \delta$~~ for $\epsilon > 0$ we choose $\delta = \epsilon$

such that $\| x - x_0 \| < \delta$

now

$$\| I(x) - I(x_0) \| = \| x - x_0 \| < \delta = \epsilon$$

so $\| I(x) - I(x_0) \| < \epsilon \qquad \forall x \in X$

so I is continuous at x_0

since x_0 was an arbitaray on X

I is continuous at X

Example

Let $T : R^2 \longrightarrow R^2$ such that

$$T(x) = T(x_1, x_2) = (x_2, x_1)$$

with $\| x \| = \| (x_1, x_2) \| = |x_1| + |x_2|$

then T is continuous and bounded

Solution Let $x, x_0 \in R^2$ such that

$x_0 = (x_{01}, x_{02})$

$x = (x_1, x_2)$

$$\|x - x_0\| = \|(x_1, x_2) - (x_{01}, x_{02})\|$$
$$= \|(x_1 - x_{01}, x_2 - x_{02})\|$$
$$= |x_1 - x_{01}| + |x_2 - x_{02}| < \delta$$

now we take $\delta = \epsilon$

$$\|T(x) - T(x_0)\| = \|T(x_1, x_2) - T(x_{01}, x_{02})\|$$
$$= \|(x_2, x_1) - (x_{02}, x_{01})\|$$
$$= \|(x_2 - x_{02}, x_1 - x_{01})\|$$
$$= |x_2 - x_{02}| + |x_1 - x_{01}| < \delta = \epsilon$$

$\therefore \|T(x) - T(x_0)\| < \epsilon \qquad \forall x \in R^2$

$\therefore T$ is continuous at x_0

Since x_0 is an arbitrary so T is continuous on R^2.

Example Let $C[a,b]$ space of all continuous function with ~~Range~~

$$\|f\| = \max_{a \le t \le b} \{|f(t)|\}$$

Let $T: C[a,b] \longrightarrow C[a,b]$ defined as follows.

(30)

$$(T(f))(t) = f(t)^2 \qquad t \in [a,b]$$
$$f \in C[a,b]$$

is T linear operator. H.w.

is T continuous.

<u>Solution</u>

Let $f_0 \in C[a,b]$ and $f \in C[a,b]$

$$M = \max_{a \le t \le b} \{ |f_0(t)| \} \text{ ----- } (1)$$

when $\| f - f_0 \| < \delta$

we choose $\delta = \min \{ 1, \frac{\epsilon}{2M+1} \}$ ——— (2)

for any $f \in C[a,b]$

$$\| T(f) - T(f_0) \| = \max_{a \le t \le b} \{ |T(f) - T(f_0)|(t)| \}$$

$$= \max_{a \le t \le b} \{ |T(f)(t) - T(f_0)(t)| \}$$

$$= \max_{a \le t \le b} \{ |f(t)^2 - f_0(t)^2| \}$$

$$= \max_{a \le t \le b} \{ |(f(t) - f_0(t))(f(t) + f_0(t))| \}$$

$$\le \max_{a \le t \le b} \{ |f(t) - f_0(t)| \, |f(t) + f_0(t)| \}$$

$$= \max_{a \le t \le b} \{ |f(t) - f_0(t)| \} \max_{a \le t \le b} |f(t) - f_0(t)| \}$$

$$\|T(f)-T(f_0)\| \leq \|f-f_0\| \max_{a\leq t\leq b} |f(t)+f_0(t)|$$

$$< \delta \max_{a\leq t\leq b} \{|f(t)-f_0(t)+f_0(t)+f_0(t)|\}$$

$$\leq \delta \max_{a\leq t\leq b} \{|f(t)-f_0(t)| + |2f_0(t)|\}$$

$$= \delta(\max_{a\leq t\leq b} |f(t)-f_0(t)| + 2\underbrace{\max_{a\leq t\leq b} |f_0(t)|}_{\text{from } ①})$$

$$< \delta(\delta+2M)$$

$$\leq \delta(1+2M)$$

$$\leq \frac{\epsilon}{1+2M}(1+2M) \longrightarrow \text{from } ②$$

$$= \epsilon$$

$\therefore$ T is continuous at f_0

Since f_0 was arbitrary it follow that T is continuous on $C[a,b]$

Example Any norm $\|\cdot\|: X \to \mathbb{R}$ on vector space X over K is continuous on X

Solution Let $x_0 \in X$ is fixed with $\|x-x_0\| < \delta$ for all $x \in X$ take $\delta = \epsilon$

$\because \big|\|x\| - \|x_0\|\big| < \|x-x_0\| < \delta = \epsilon$

$\therefore \big|\|x\| - \|x_0\|\big| < \epsilon \quad \forall x \in X$

$\therefore \|\cdot\|$ is continuous at x_0, since x_0 was arbitrary so $\|\cdot\|$ is continuous at X.

31 1 Lemma 8 The addition operator

$+ : X \times X \longrightarrow X$ when $(X, \|\cdot\|)$ is normed

when $(X \times X, \|\cdot\|_2)$ also normed

$\| (x,y) \|_2 = \|x\| + \|y\|$

proof: let $z_0 = (x_0, y_0) \in X \times X$ be fixed

such for $z = (x,y)$ and any $\delta > 0$ $\|z - z_0\|_2 < \delta$

$$\|z - z_0\|_2 = \| (x,y) - (x_0, y_0) \|$$
$$= \| (x - x_0), (y - y_0) \|$$
$$= \|x - x_0\| + \|y - y_0\| < \delta$$

we take $\delta = \epsilon$

$$\| +(z) - +(z_0) \| = \| +(x,y) - +(x_0,y_0) \|$$
$$= \| x+y - (x_0 + y_0) \|$$
$$= \| x - x_0 + y - y_0 \|$$
$$\leq \|x - x_0\| + \|y - y_0\|$$
$$< \delta = \epsilon$$

$\therefore$ + is continuous at $z_0 = (x_0, y_0)$

since z_0 is an arbitrary on $X \times X$

So + is continuous at $X \times X$.