2.2 Hilbert-Adjoint Operator.

2.1 Definition (Hilbert adjoint operator)
Let $T: H_1 \longrightarrow H_2$ be a bounded linear operator, where H_1, H_2 are Hilbert spaces. Then the Hilbert-adjoint operator T^* of T is the operator $T^*: H_2 \longrightarrow H_1$ such that for $x \in H_1$ and $y \in H_2$.

$$\langle Tx / y \rangle = \langle x / T^* y \rangle.$$

Example (1) ~~Let $I: ¢^3 \to ¢^3$ be the identity~~ operator we define $T: ¢^2 \longrightarrow ¢^2$ such that $T(x) = ix$ where $x \in ¢^2$
find T^* operator T is bonded linear operator explain in page 5

Solution $T^*: ¢^2 \longrightarrow ¢^2$ such that

$$\langle T(x) / y \rangle = \langle x / T^*(y) \rangle$$

$$\langle ix / y \rangle = \langle x / T^*(y) \rangle$$

$$i \langle x / y \rangle = \langle x / T^*(y) \rangle$$

$$\langle x / \bar{i} y \rangle = \langle x / T^*(y) \rangle$$

$$\langle x / -iy \rangle = \langle x / T^*(y) \rangle$$

$\therefore T^*(y) = -iy$ is adjoint operator of T

~~Example (2) Let $T: \mathbb{C}^2 \to \mathbb{C}^2$ such that~~

~~$T(x) = T(x_1, x_2) = \sqrt{2}\, i (x_2, x_1)$~~

~~Find T^*~~

~~Solution $T^*: \mathbb{C}^2 \to \mathbb{C}^2$~~

Theorem (Existence) The Hilbert-adjoint operator T^* of T exists, is unique and is a bounded linear operator with norm

$$\|T^*\| = \|T\|$$

proof: we define a sesquilinear from $H_2 \times H_1$ ~~$h(y,x) = \langle y$~~ $h: H_2 \times H_1 \to K$ such that

$$h(y, x) = \langle y, T(x) \rangle \longrightarrow (1)$$

because the inner product it is sesqulinear and T is linear. It is conjugate linear in sesquilinear form.

$$h(y, \alpha x_1 + \beta x_2) = \langle y, T(\alpha x_1 + \beta x_2) \rangle$$
$$= \langle y, \alpha T(x_1) + \beta T(x_2) \rangle$$
$$= \bar{\alpha} \langle y, T(x_1) \rangle + \bar{\beta} \langle y, T(x_2) \rangle$$
$$= \bar{\alpha}\, h(y, x_1) + \bar{\beta}\, h(y, x_2)$$

$$h(\alpha y_1 + \beta y_2, x) = \langle \alpha y_1 + \beta y_2, T(x) \rangle$$
$$= \alpha \langle y_1, T(x) \rangle + \beta \langle y_2, T(x) \rangle$$
$$= \alpha\, h(y_1, x) + \beta\, h(y_2, x)$$

this proof that h is sesquilinear form

to show h is bounded

$$|h(y,x)| = |\langle y, T(x)\rangle|$$

$$\leq \|y\| \, \|T(x)\|$$

by using ~~St~~ Schwarz inequality

$$= \|y\| \, \|T\| \, \|x\|$$

since T is bounded.

$\therefore$ h is bounded and

$$\|h\| = \sup_{\substack{0 \neq y \in H_2 \\ 0 \neq x \in H_1}} \left\{ \frac{|h(y,x)|}{\|x\| \, \|y\|} \right\}$$

$$\leq \sup_{\substack{0 \neq y \in H_2 \\ 0 \neq x \in H_1}} \left\{ \frac{\cancel{\|y\|} \, \|T\| \, \cancel{\|x\|}}{\cancel{\|y\|} \, \cancel{\|x\|}} \right\}$$

$$\therefore \|h\| \leq \|T\|$$

now take $y = T(x)$ we get

$$\|h\| = \sup_{\substack{0 \neq y \in H_2 \\ 0 \neq x \in H_1}} \left\{ \frac{|h(y,x)|}{\|x\| \, \|y\|} \right\}$$

$$\geq \sup_{0 \neq x \in H_1} \left\{ \frac{|h(T(x), x)|}{\|x\| \, \|T(x)\|} \right\}$$

$$= \sup_{0 \neq x \in H_1} \left\{ \frac{|\langle T(x), T(x)\rangle|}{\|x\| \, \|T(x)\|} \right\}$$

$$= \sup_{0 \neq x \in H_1} \left\{ \frac{\|T(x)\|^2}{\|x\| \, \|T(x)\|} \right\}$$

$$\|h\| \geq \sup_{0 \neq x \in H_1} \left\{ \frac{\|T(x)\|}{\|x\|} \right\}$$

$$= \|T\|$$

we get that $\|h\| = \|T\|$

we ~~can write~~ define $h(y, x) = \langle T^*(y) / x \rangle$ — ②

from the (Theorem Riesz representation)

we get that $T^* : H_2 \to H_1$ is uniquely determined and bounded linear operator with norm $\|h\| = \|T^*\|$

now $\|T^*\| = \|h\| = \|T\|$

So $\|T\| = \|T^*\|$

from the equality ① and ②

$h(y, x) = \langle x / T(y) \rangle$

$h(y, x) = \langle T^*(x) / y \rangle$

we get that $\langle x / T(y) \rangle = \langle T^*(x) / y \rangle$

take the conjugates for both side

$$\langle T(y) / x \rangle = \overline{\langle x / T(y) \rangle} = \overline{\langle T^*(x) / y \rangle}$$

$$= \langle y / T^*(x) \rangle$$

$\therefore \langle T(y) / x \rangle = \langle y / T^*(x) \rangle$

$\therefore T^*$ is the adjoint operator of T

Example we see in example (1)

for $T: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ $T(x) = ix$

the adjoint operator $T^*: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is exists $T^*(\cancel{x}) = -i\cancel{x}$

where T is bounded linear operator, for $\alpha, \beta \in \mathbb{C}$, $x, y \in \mathbb{C}^2$

$$T(\alpha x + \beta y) = i(\alpha x + \beta y)$$
$$= \alpha(ix) + \beta(iy)$$
$$= \alpha T(x) + \beta T(y)$$

$\therefore$ T is linear

$$\|T(x)\| = \|ix\|$$
$$= |i|\|x\|$$
$$= \|x\|$$ So T is bounded linear operator

$$\|T\| = \sup_{0 \neq x \in \mathbb{C}^2} \left\{ \frac{\|T(x)\|}{\|x\|} \right\} = \sup_{0 \neq x \in \mathbb{C}^2} \left\{ \frac{\|x\|}{\|x\|} \right\} = 1$$

$\therefore$ $\|T\| = 2$

T^* is also bounded linear operator $\cancel{v}, w \in \mathbb{C}^2$, $\alpha, \beta \in \mathbb{C}$

$$T^*(\alpha v + \beta w) = -i(\alpha v + \beta w)$$
$$= -i\alpha v + -i\beta w$$
$$= \alpha(-iv) + \beta(-iw)$$
$$= \alpha T^*(v) + \beta T^*(w)$$

$\therefore$ T^* is linear operator

To show that T^* is bounded

$$\|T^*(w)\| = \|-iw\|$$
$$\leq |-i|\,\|w\|$$
$$\leq \|w\|$$

$\cancel{\|T^*\|} \leq$

$$\|T^*\| \leq \sup_{0 \neq w \in \mathbb{C}^1}\left\{\frac{\|T^*(w)\|}{\|w\|}\right\} \leq \sup_{0 \neq w \in \mathbb{C}^1}\left\{\frac{\|w\|}{\|w\|}\right\} \leq 1 = 1$$

So $\|T\| \leq 1 \leq \|T^*\|$

Lemma (zero operator)

Let X and Y be pre Hilbert space and $T: X \to Y$ be a bounded linear operator. Then

(1) $T = 0$ (zero operator) if and only if $\langle T(x)/y \rangle = 0$ $\forall x \in X, y \in X$.

(2) if $T: X \to X$ where X is complex vector space and $\langle T(x)/x \rangle = 0$ $\forall x \in X$ then $T = 0$

proof: (1)

Let $T = 0$ this implies that $T(x) = 0$ $\forall x \in X$

So $\langle T(x)/y \rangle = \langle 0/y \rangle = \langle 0 \cdot w/y \rangle = 0 \langle w/y \rangle$
$= 0$

Conversely

Let $\langle T(x)/y\rangle = 0 \quad \forall x \in X, y \in Y$

Let $y = T(x)$, then we get

$\langle T(x)/T(x)\rangle = 0$, so

$\|T(x)\|^2 = 0$

$\|T(x)\| = 0$

$T(x) = 0 \quad \forall x \in X$

$\therefore T = 0.$

(2) $\exists$ $T: X \longrightarrow X$ such that

~~$\langle T(x)/x\rangle$~~ $= 0 \quad \forall x \in X$

$\langle T(v)/v\rangle = 0$

Let $v = \alpha x + y \in X$ wher $x, y \in X$
$\alpha \in \mathbb{C}$

now

$0 = \langle T(v)/v\rangle = \langle T(\alpha x+y)/\alpha x+y\rangle$

$= \langle \alpha T(x)+T(y)/\alpha x+y\rangle$

$= \langle \alpha T(x)/\alpha x\rangle + \langle \alpha T(x)/y\rangle + \langle T(y)/\alpha x\rangle + \langle T(y)/y\rangle$

$= \alpha\bar{\alpha}\underbrace{\langle T(x)/x\rangle}_{=0} + \alpha\langle T(x)/y\rangle + \bar{\alpha}\langle T(y)/x\rangle + \underbrace{\langle T(y)/y\rangle}_{=0}$

$\therefore \alpha\langle T(x)/y\rangle + \bar{\alpha}\langle T(y)/x\rangle = 0$

set $\alpha = 1$ we get

$\langle T(x)/y\rangle + \langle T(y)/x\rangle = 0 \quad$ — (1)

now set $\alpha = i$, then $\bar{\alpha} = -i$

$$i\langle T(x)|y\rangle - i\langle T(y)|x\rangle = 0 \quad \text{— (2)}$$

we get that

$$\langle T(x)|y\rangle + \langle T(y)|x\rangle = 0$$

$$i\langle T(x)|y\rangle - i\langle T(y)|x\rangle = 0$$

$$i\langle T(x)|y\rangle + \cancel{i\langle T(y)|x\rangle} = 0$$

$$i\langle T(x)|y\rangle - \cancel{i\langle T(y)|x\rangle} = 0 \qquad \text{بالجمع}$$

$$\frac{2i\langle T(x)|y\rangle}{2i} = \frac{0}{2i}$$

$$\langle T(x)|y\rangle = 0 \quad \forall x \in X,\ y \in X$$

~~set $y = T(x)$~~

~~$\langle T(x)|T(x)\rangle$~~

from the first part (1) we get that

$T = 0$.

Note The second prove is true when X is vector space over complex number and it is not true when X be vector space over field of real number