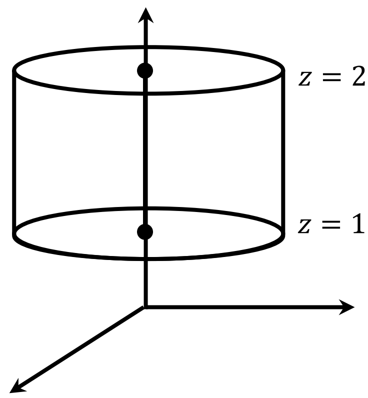
The Divergence Theorem
نــظــريـــة التـــبـــاعــــــد

Theorem: Let $F(x,y,z) = f(x,y,z)i + g(x,y,z)j + h(x,y,z)k$ is a vector which is defined and continuously differentiable in a bounded closed region G. Let S be the surface of G, and n the unit outward normal vector to S . Then

$$\iint_S F.n.ds = \int\int_G\int \text{div}\, F\, dV$$

Example 1: Use divergence theorem to find the outward Flux of the vector field $F(x,y,z) = x^3 i + y^3 j + z^2 k$ across (يعبر) the cylinder $x^2 + y^2 = 9$, bounded below $z = 2$ above $z = 1$.



Solution:

$$\text{div} = 3x^2 + 3y^2 + 2z$$

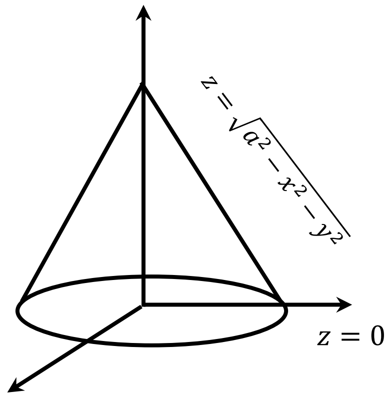
$$\int\int_G\int (3x^2 + 3y^2 + 2z)\, dV =$$

$$\int_0^{2\pi}\int_0^3\int_1^2 (3r^2 + 2z)\ r dz\, dr\, d\theta =$$

$$\int_0^{2\pi}\int_0^3 (3r^2 z + z^2)\Big]_1^2\ r\, dr\, d\theta = \int_0^{2\pi}\int_0^3 \left[(6r^2 + 4) - (3r^2 + 1)\right]\ r\, dr\, d\theta =$$

$$= \int_0^{2\pi}\int_0^3 (3r^2 + 3)\ r\, dr\, d\theta = \int_0^{2\pi}\int_0^3 (3r^3 + 3r)\ dr\, d\theta = \int_0^{2\pi} \frac{3r^4}{4} + 3\frac{r^2}{2}\Big]_0^3 d\theta$$

Example 2: Use divergence theorem to find the outward Flux of the vector field $F(x,y,z) = x^3 i + y^3 j + z^3 k$ enclosed by semi-spherical $z = \sqrt{a^2 - x^2 - y^2}$, and the plane $z = 0$.



Solution:

div $= 3x^2 + 3y^2 + 3z^2$

$$\iiint_G (3x^2 + 3y^2 + 3z^2)\, dV =$$

$$\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{a} 3\rho^2 \ \rho^2 \, sin\emptyset \, d\rho \, d\emptyset \, d\theta =$$

$$\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \frac{3\rho^5}{5}\bigg]_0^a \ sin\emptyset \ d\emptyset \, d\theta = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \frac{3a^5}{5} \, sin\emptyset \ d\emptyset \, d\theta =$$

$$= -\int_0^{2\pi} \frac{3a^5}{5} \cos\emptyset\bigg]_0^{\frac{\pi}{2}} d\theta = -\frac{3a^5}{5}\int_0^{2\pi} (-1)\, d\theta = \frac{6a^5\pi}{5}$$
