Answers

1. (*i*) $\frac{1}{6}[e^{2t} - e^{-4t}]$ (*ii*) $\frac{2}{5}[e^{3t} - e^{-7t}]$

2. (*i*) $e^t - t - 1$ (*ii*) $1 - \cos t$

3. (*i*) $(t + 2)e^{-t} + t - 2$ (*ii*) $\cos t + \frac{t^2}{2} - 1$

4. (*i*) $\frac{1}{2}[e^{-t} + \sin t - \cos t]$ (*ii*) $\frac{1}{2}(\sin t - t\cos t)$

5. (*i*) $\frac{1}{a^3}[\sinh at - at]$ (*ii*) $1 - e^{-at}(1 + at)$

6. (*i*) $-\frac{1}{e^t}\left[\frac{t^2}{2} + t + 1\right] + 1$ (*ii*) $2[e^{-t} + \sin t - \cos t]$

7. (*i*) $\frac{1}{2}\left[t\cos 2t + \frac{1}{2}\sin 2t\right]$ (*ii*) $\frac{1}{2}\left[t\cosh 3t + \frac{1}{3}\sinh 3t\right]$

8. (*i*) $\frac{1}{3a^2}\left[e^{at} - e^{-\frac{at}{2}}\left(\cos\frac{\sqrt{3}}{2}at + \sqrt{3}\sin\frac{\sqrt{3}}{2}at\right)\right]$ (*ii*) $\frac{1}{2}te^{-2t}\sin t$

9. (*i*) $-\frac{1}{4}[2te^{-2t} + e^{-2t} - 1]$ (*ii*) $-\frac{1}{8e^{2t}}(2t^2 + 2t + 1) + \frac{1}{8}$.

2.13. INVERSE LAPLACE TRANSFORMS BY THE METHOD OF PARTIAL FRACTIONS

Let $\frac{a_0s^m + a_1s^{m-1} + \ldots\ldots + a_m}{b_0s^n + b_1s^{n-1} + \ldots\ldots + b_n}$ be a proper rational algebraic function of s with $m < n$. The denominator of this quotient can be factorised into linear and quadratic factors. The given rational function can be expressed as the sum of partial fractions as per the rules given below :

(*i*) If $as + b$ is any linear non-repeated factor in the denominator, then there corresponds a partial fraction of the form $\frac{A}{as + b}$.

(*ii*) If $as + b$ is any linear factor repeated r ($\in \mathbf{N}$) times in the denominator, then there corresponds partial fractions of the form $\frac{A}{as + b}, \frac{B}{(as + b)^2}, \frac{C}{(as + b)^3}, \ldots\ldots r$ terms.

(*iii*) If $as^2 + bs + c$ is any irreducible quadratic factor in the denominator, then there corresponds partial fraction of the form $\frac{As + B}{as^2 + bs + c}$.

(*iv*) If $as^2 + bs + c$ is any irreducible factor repeated r ($\in \mathbf{N}$) times in the denominator, then there corresponds partial fractions of the form $\frac{As + B}{as^2 + bs + c}, \frac{Cs + D}{(as^2 + bs + c)^2}, \ldots\ldots r$ terms.

The quantities A, B, C, D, ... are all constants independent of s.

The constants A, B, C, D, ... occurring in the numerators of the partial fractions are determined by simplifying the sum of partial fractions and then giving various values to s, to obtain equations involving unknown constants or by comparing the coefficients of like powers of s.

$\therefore$ The given proper fraction can be expressed as the sum of its partial fractions. Thus, by using linearity of inverse Laplace transform and elementary inverse Laplace transform formulae, the inverse Laplace transform of the given proper fraction is found.

If the given fraction is not proper, then division of numerator by denominator is carried first.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the inverse Laplace transform of the following functions* :

(*i*) $\dfrac{s^2+s-2}{s(s+3)(s-2)}$ (*ii*) $\dfrac{s^2-10s+13}{(s-7)(s^2-5s+6)}$.

Sol. (*i*) $\dfrac{s^2+s-2}{s(s+3)(s-2)}$ is a proper fraction.

Let
$$\frac{s^2+s-2}{s(s+3)(s-2)}=\frac{A}{s}+\frac{B}{s+3}+\frac{C}{s-2}$$

Multiplying both sides by $s(s+3)(s-2)$, we get

$$s^2+s-2=A(s+3)(s-2)+Bs(s-2)+Cs(s+3) \quad ...(1)$$

$s=0$ in (1) $\Rightarrow$ $-2=A(3)(-2)+0+0$ $\Rightarrow$ $A=-2/-6=1/3$

$s=-3$ in (1) $\Rightarrow$ $9-3-2=0+B(-3)(-5)+0$ $\Rightarrow$ $B=4/15$

$s=2$ in (1) $\Rightarrow$ $4+2-2=0+0+C(2)(5)$ $\Rightarrow$ $C=4/10=2/5$

$$\therefore \quad \frac{s^2+s-2}{s(s+3)(s-2)}=\frac{1/3}{s}+\frac{4/15}{s+3}+\frac{2/5}{s-2}=\frac{1}{3}\left(\frac{1}{s}\right)+\frac{4}{15}\left(\frac{1}{s+3}\right)+\frac{2}{5}\left(\frac{1}{s-2}\right)$$

$$\therefore \quad L^{-1}\left(\frac{s^2+s-2}{s(s+3)(s-2)}\right)=\frac{1}{3}L^{-1}\left(\frac{1}{s}\right)+\frac{4}{15}L^{-1}\left(\frac{1}{s+3}\right)+\frac{2}{5}L^{-1}\left(\frac{1}{s-2}\right)$$

$$=\frac{1}{3}(1)+\frac{4}{15}e^{-3t}+\frac{2}{5}e^{2t}=\boldsymbol{\frac{1}{3}+\frac{4}{15}e^{-3t}+\frac{2}{5}e^{2t}}.$$

Note. Since the factors in the denominator are linear and non-repeated, the short-cut method can also be used.

(*ii*) $\dfrac{s^2 - 10s + 13}{(s-7)(s^2 - 5s + 6)} = \dfrac{s^2 - 10s + 13}{(s-7)(s-2)(s-3)}$ is a proper fraction.

The factors in the denominator are linear and non-repeated.

$\therefore$ By short-cut method,

$$\frac{s^2 - 10s + 13}{(s-7)\,(s^2 - 5s + 6)} = \frac{7^2 + 10(7) + 13}{(\text{s}-7)\,(5)\,(4)} + \frac{2^2 - 10(2) + 13}{(-5)\,(s-2)\,(-1)} + \frac{3^2 - 10(3) + 13}{(-4)\,(1)\,(s-3)}$$

$$= \frac{-8}{20\,(s-7)} + \frac{-3}{5\,(s-2)} + \frac{-8}{-4\,(s-3)}$$

$$= -\frac{2}{5\,(s-7)} - \frac{3}{5\,(s-2)} + 2\,.\,\frac{1}{s-3}$$

$$\therefore \quad \mathrm{L}^{-1}\left(\frac{s^2 - 10s + 13}{(s-7)\,(s^2 - 5s + 6)}\right) = \mathrm{L}^{-1}\left(-\frac{2}{5\,(s-7)} - \frac{3}{5\,(s-2)} + 2\,.\,\frac{1}{s-3}\right)$$

$$= -\frac{2}{5}\,\mathrm{L}^{-1}\left(\frac{1}{s-7}\right) - \frac{3}{5}\,\mathrm{L}^{-1}\left(\frac{1}{s-2}\right) + 2\mathrm{L}^{-1}\left(\frac{1}{s-3}\right)$$

$$= -\boldsymbol{\frac{2}{5}\,\mathrm{e}^{7t} - \frac{3}{5}\,\mathrm{e}^{2t} + 2\mathrm{e}^{3t}}.$$

Example 2. *Find the inverse Laplace transform of the function* $\dfrac{s^2}{(s^2 + a^2)\,(s^2 + b^2)}$ *by using (i) method of partial fractions (ii) convolution theorem.*

Sol. (*i*) $\dfrac{s^2}{(s^2 + a^2)\,(s^2 + b^2)} = \dfrac{z}{(z + a^2)\,(z + b^2)}$, where $z = s^2$

$$= \frac{-a^2}{(z + a^2)\,(-a^2 + b^2)} + \frac{-b^2}{(-b^2 + a^2)\,(z + b^2)}$$

$$= \frac{a^2}{a^2 - b^2}\,.\,\frac{1}{z + a^2} - \frac{b^2}{a^2 - b^2}\,.\,\frac{1}{z + b^2}$$

$$= \frac{a}{a^2 - b^2}\,.\,\frac{a}{s^2 + a^2} - \frac{b}{a^2 - b^2}\,.\,\frac{b}{s^2 + b^2}\,.$$

$$\therefore \quad \mathrm{L}^{-1}\left(\frac{s^2}{(s^2 + a^2)\,(s^2 + b^2)}\right) = \frac{a}{a^2 - b^2}\,\mathrm{L}^{-1}\left(\frac{a}{s^2 + a^2}\right) - \frac{b}{a^2 - b^2}\,\mathrm{L}^{-1}\left(\frac{b}{s^2 + b^2}\right)$$

$$= \frac{a}{a^2 - b^2} \sin at - \frac{b}{a^2 - b^2} \sin bt$$

$$= \frac{\mathbf{a \sin at - b \sin bt}}{\mathbf{a^2 - b^2}}.$$

(*ii*) $$\frac{s^2}{(s^2 + a^2)(s^2 + b^2)} = \left(\frac{s}{s^2 + a^2} \right) \left(\frac{s}{s^2 + b^2} \right) = \mathrm{F}(s)\ \mathrm{G}(s),$$

where $\mathrm{F}(s) = \dfrac{s}{s^2 + a^2}$ and $\mathrm{G}(s) = \dfrac{s}{s^2 + b^2}$.

Also $$\mathrm{L}^{-1} (\mathrm{F}(s)) = \mathrm{L}^{-1} \left(\frac{s}{s^2 + a^2} \right) = \cos at = f(t), \text{ say}$$

and $$\mathrm{L}^{-1} (\mathrm{G}(s)) = \mathrm{L}^{-1} \left(\frac{s}{s^2 + b^2} \right) = \cos bt = g(t), \text{ say.}$$

By **convolution theorem,** we have

$$\mathrm{L}^{-1} (\mathrm{F}(s)\ \mathrm{G}(s)) = f * g = \int_0^t f(\mathrm{T})\ g(t - \mathrm{T})\ d\mathrm{T}.$$

$$\therefore \quad \mathrm{L}^{-1} \left(\frac{s^2}{(s^2 + a^2)(s^2 + b^2)} \right) = \int_0^t \cos a\mathrm{T} \cos b(t - \mathrm{T})\, d\mathrm{T}$$

$$= \frac{1}{2} \int_0^t [\cos (a\mathrm{T} + bt - b\mathrm{T}) + \cos (a\mathrm{T} - bt + b\mathrm{T})]\, d\mathrm{T}$$

$$= \frac{1}{2} \left[\frac{\sin ((a - b)\,\mathrm{T} + bt)}{a - b} + \frac{\sin ((a + b)\,\mathrm{T} - bt)}{a + b} \right]_0^{\mathrm{t}}$$

$$= \frac{1}{2} \left[\frac{\sin at}{a - b} + \frac{\sin at}{a + b} - \frac{\sin bt}{a - b} + \frac{\sin bt}{a + b} \right]$$

$$= \frac{1}{2} \left[\frac{2a}{a^2 - b^2} \sin at - \frac{2b}{a^2 - b^2} \sin bt \right]$$

$$= \frac{\mathbf{a \sin at - b \sin bt}}{\mathbf{a^2 - b^2}}.$$

Example 3. *Find the inverse Laplace transform of the function* $\dfrac{1}{(s - 1)^5\ (s + 2)}$ *by using (i) method of partial fractions (ii) convolution theorem.*

Sol. (*i*) $$\frac{1}{(s - 1)^5\ (s + 2)} = \frac{1}{z^5\ (z + 3)}, \text{ where } \quad z = s - 1.$$

We divide 1 by $z + 3$ as follows :

$$
\begin{array}{r|l}
 & \dfrac{1}{3}-\dfrac{z}{9}+\dfrac{z^2}{27}-\dfrac{z^3}{81}+\dfrac{z^4}{243} \\ \hline
3+z & 1 \\
 & 1+\dfrac{z}{3} \\ \hline
 & -\dfrac{z}{3} \\
 & -\dfrac{z}{3}-\dfrac{z^2}{9} \\ \hline
 & \dfrac{z^2}{9} \\
 & \dfrac{z^2}{9}+\dfrac{z^3}{27} \\ \hline
 & -\dfrac{z^3}{27} \\
 & -\dfrac{z^3}{27}-\dfrac{z^4}{81} \\ \hline
 & \dfrac{z^4}{81} \\
 & \dfrac{z^4}{81}+\dfrac{z^5}{243} \\ \hline
 & -\dfrac{z^5}{243} \\ \hline
\end{array}
$$

$$\therefore \quad \frac{1}{3+z}=\frac{1}{3}-\frac{z}{9}+\frac{z^2}{27}-\frac{z^3}{81}+\frac{z^4}{243}-\frac{z^5/243}{3+z}$$

$$\therefore \quad \frac{1}{z^5(z+3)}=\frac{1}{z^5}\left(\frac{1}{3+z}\right)=\frac{1}{3z^5}-\frac{1}{9z^4}+\frac{1}{27z^3}-\frac{1}{81z^2}+\frac{1}{243z}-\frac{1}{243(3+z)}$$

$$\therefore \quad \frac{1}{(s-1)^5\ (s+2)}=\frac{1}{3(s-1)^5}-\frac{1}{9(s-1)^4}+\frac{1}{27(s-1)^3}$$

$$-\frac{1}{81(s-1)^2}+\frac{1}{243(s-1)}-\frac{1}{243(s+2)}$$

$$\therefore \quad L^{-1}\left(\frac{1}{(s-1)^5(s+2)}\right)=\frac{1}{3}L^{-1}\left(\frac{1}{(s-1)^5}\right)-\frac{1}{9}L^{-1}\left(\frac{1}{(s-1)^4}\right)$$

$$+\frac{1}{27}L^{-1}\left(\frac{1}{(s-1)^3}\right)-\frac{1}{81}L^{-1}\left(\frac{1}{(s-1)^2}\right)+\frac{1}{243}L^{-1}\left(\frac{1}{s-1}\right)-\frac{1}{243}L^{-1}\left(\frac{1}{s+2}\right)$$

$$= \frac{1}{3} e^t \, L^{-1}\left(\frac{1}{s^5}\right) - \frac{1}{9} e^t \, L^{-1}\left(\frac{1}{s^4}\right) + \frac{1}{27} e^t \, L^{-1}\left(\frac{1}{s^3}\right)$$

$$- \frac{1}{81} e^t \, L^{-1}\left(\frac{1}{s^2}\right) + \frac{1}{243} e^t \, L^{-1}\left(\frac{1}{s}\right) - \frac{1}{243} e^{-2t} \, L^{-1}\left(\frac{1}{s}\right)$$

$$= \frac{e^t}{3} \cdot \frac{1}{24} L^{-1}\left(\frac{4!}{s^5}\right) - \frac{e^t}{9} \cdot \frac{1}{6} L^{-1}\left(\frac{3!}{s^4}\right) - \frac{e^t}{27} \cdot \frac{1}{2} L^{-1}\left(\frac{2!}{s^3}\right) - \frac{e^t}{81} \cdot t + \frac{e^t}{243} \cdot 1 - \frac{e^{-2t}}{243} \cdot 1$$

$$= \mathbf{\frac{e^t}{72} t^4 - \frac{e^t}{54} t^3 - \frac{e^t}{54} t^2 - \frac{e^t}{81} t + \frac{e^t}{243} - \frac{e^{-2t}}{243}}.$$

(*ii*) $$\frac{1}{(s-1)^5 (s+2)} = \left(\frac{1}{(s-1)^5}\right)\left(\frac{1}{s+2}\right) = F(s) \, G(s),$$

where $$F(s) = \frac{1}{(s-1)^5} \quad \text{and} \quad G(s) = \frac{1}{s+2}.$$

Also $$L^{-1}(F(s)) = L^{-1}\left(\frac{1}{(s-1)^5}\right) = e^{1.t} \, L^{-1}\left(\frac{1}{s^5}\right)$$

$$= e^t \cdot \frac{1}{24} L^{-1}\left(\frac{4!}{s^5}\right) = \frac{e^t}{24} t^4 = f(t), \text{ say}$$

and $$L^{-1}(G(s)) = L^{-1}\left(\frac{1}{s+2}\right) = e^{-2t} = g(t), \text{ say}.$$

By **convolution theorem,** we have

$$L^{-1}(F(s) \, G(s)) = f * g = \int_0^t f(T) \, g(t-T) \, dT.$$

$$\therefore \quad L^{-1}\left(\frac{1}{(s-1)^5 (s+2)}\right) = \int_0^t \frac{e^T T^4}{24} \cdot e^{-2(t-T)} \, dT = \frac{e^{-2t}}{24} \int_0^t T^4 e^{3T} \, dT.$$

Now $$\int_0^t T^4 e^{3T} \, dT = \frac{T^4 e^{3T}}{3}\Bigg|_0^t - \int_0^t 4T^3 \cdot \frac{e^{3T}}{3} \, dT$$

$$= \frac{t^4 e^{3t}}{3} - \frac{4}{3}\left[\frac{T^3 e^{3T}}{3}\Bigg|_0^t - \int_0^t 3T^2 \cdot \frac{e^{3T}}{3} \, dT\right]$$

$$= \frac{t^4 e^{3t}}{3} - \frac{4}{9} t^3 e^{3t} + \frac{4}{3}\left[\frac{T^2 e^{3T}}{3}\Bigg|_0^t - \int_0^t 2T \cdot \frac{e^{3T}}{3} \, dT\right]$$

$$= \frac{t^4 e^{3t}}{3} - \frac{4}{9} t^3 e^{3t} + \frac{4}{9} t^2 e^{3t} - \frac{8}{9}\left[\frac{T e^{3T}}{3}\Bigg|_0^t - \int_0^t 1 \cdot \frac{e^{3T}}{3} \, dT\right]$$

$$= \frac{t^4 e^{3t}}{3} - \frac{4}{9} t^3 e^{3t} + \frac{4}{9} t^2 e^{3t} - \frac{8}{27} t e^{3t} + \frac{8}{81} (e^{3t} - 1)$$

$$\therefore \quad L^{-1}\left(\frac{1}{(s-1)^5 (s+2)}\right) = \frac{e^{-2t}}{24}\left[\frac{t^4 e^{3t}}{3} - \frac{4}{9} t^3 e^{3t} + \frac{4}{9} t^2 e^{3t} - \frac{8}{27} t e^{3t} + \frac{8}{81} e^{3t} - \frac{8}{81}\right]$$

$$= \mathbf{\frac{t^4 e^t}{72} - \frac{t^3 e^t}{54} + \frac{t^2 e^t}{54} - \frac{t e^t}{81} + \frac{e^t}{243} - \frac{e^{-2t}}{243}}.$$

Example 4. *Find the inverse Laplace transform of the following functions :*

(*i*) $\dfrac{4s+5}{(s-1)^2 (s+2)}$ (*ii*) $\dfrac{1+2s}{(s+2)^2 (s-1)^2}$.

Sol. (*i*) $\dfrac{4s+5}{(s-1)^2 (s+2)}$ is a proper fraction.

Let $\dfrac{4s+5}{(s-1)^2 (s+2)} = \dfrac{A}{s-1} + \dfrac{B}{(s-1)^2} + \dfrac{C}{s+2}$

Multiplying both sides by $(s-1)^2 (s+2)$, we get

$$4s + 5 = A(s-1)(s+2) + B(s+2) + C(s-1)^2. \qquad ...(1)$$

$s = 1$ in (1) $\Rightarrow$ $9 = 0 + 3B + 0$ $\Rightarrow$ $B = 3$

$s = -2$ in (1) $\Rightarrow$ $-3 = 0 + 0 + 9C$ $\Rightarrow$ $C = -1/3$

Let $s = 0$. $\therefore$ $5 = -2A + 2B + C$

$\Rightarrow$ $2A = 2(3) - (1/3) - 5 = 2/3$ $\Rightarrow$ $A = 1/3$.

$$\therefore \quad \frac{4s+5}{(s-1)^2 (s+2)} = \frac{1/3}{s-1} + \frac{3}{(s-1)^2} + \frac{-1/3}{s+2} = \frac{1}{3}\left(\frac{1}{s-1}\right) + 3\left(\frac{1}{(s-1)^2}\right) - \frac{1}{3}\left(\frac{1}{s+2}\right)$$

$$\therefore \quad L^{-1}\left(\frac{4s+5}{(s-1)^2 (s+2)}\right) = \frac{1}{3} L^{-1}\left(\frac{1}{s-1}\right) + 3L^{-1}\left(\frac{1}{(s-1)^2}\right) - \frac{1}{3} L^{-1}\left(\frac{1}{s+2}\right)$$

$$= \frac{1}{3} e^{1.t} + 3e^{1.t} L^{-1}\left(\frac{1}{s^2}\right) - \frac{1}{3} e^{-2t}$$

$$= \frac{1}{3} e^t + 3e^t t^1 - \frac{1}{3} e^{-2t} = \mathbf{\frac{1}{3}((9t+1)\, e^t - e^{-2t})}.$$

(*ii*) $\dfrac{1+2s}{(s+2)^2 (s-1)^2}$ is a proper fraction.

Let $\dfrac{1+2s}{(s+2)^2 (s-1)^2} = \dfrac{A}{s+2} + \dfrac{B}{(s+2)^2} + \dfrac{C}{s-1} + \dfrac{D}{(s-1)^2}$

Multiplying both sides by $(s+2)^2 (s-1)^2$, we get

$$1 + 2s = A(s+2)(s-1)^2 + B(s-1)^2 + C(s+2)^2 (s-1) + D(s+2)^2. \qquad ...(1)$$

$s = 1$ in (1) $\Rightarrow$ $3 = 0 + 0 + 0 + 9D$ $\Rightarrow$ $D = 1/3$

$s = -2$ in (1) $\Rightarrow$ $-3 = 0 + 9B + 0 + 0$ $\Rightarrow$ $B = -1/3$

Comparing coefficients of s^3 and s^2 in (1), we get

$$0 = A + C \quad ...(2) \qquad \text{and} \qquad 0 = B + 3C + D \quad ...(3)$$

$$(3) \Rightarrow 3C = -B - D = \frac{1}{3} - \frac{1}{3} = 0 \Rightarrow C = 0$$

$\therefore$ (2) $\Rightarrow$ A = – C = 0.

$$\therefore \quad \frac{1+2s}{(s+2)^2\,(s-1)^2} = \frac{0}{s+2} + \frac{-1/3}{(s+2)^2} + \frac{0}{s-1} + \frac{1/3}{(s-1)^2} = -\frac{1}{3}\left(\frac{1}{(s+2)^2}\right) + \frac{1}{3}\left(\frac{1}{(s-1)^2}\right)$$

$$\therefore \quad L^{-1}\left(\frac{1+2s}{(s+2)^2\,(s-1)^2}\right) = -\frac{1}{3} L^{-1}\left(\frac{1}{(s+2)^2}\right) + \frac{1}{3} L^{-1}\left(\frac{1}{(s-1)^2}\right)$$

$$= -\frac{1}{3} e^{-2t} L^{-1}\left(\frac{1}{s^2}\right) + \frac{1}{3} e^{1.t} L^{-1}\left(\frac{1}{s^2}\right)$$

$$= -\frac{1}{3} e^{-2t} . t + \frac{1}{3} e^{t} . t = \mathbf{\frac{t}{3}(e^t - e^{-2t}).}$$

Example 5. *Find the inverse Laplace transform of the following functions :*

(*i*) $\dfrac{3s+1}{(s+1)(s^2+1)}$ (*ii*) $\dfrac{2s^3+2s^2+4s+1}{(s^2+1)(s^2+s+1)}$.

Sol. (*i*) $\dfrac{3s+1}{(s+1)(s^2+1)}$ is a proper fraction.

Let $$\frac{3s+1}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}.$$

Multiplying both sides by $(s + 1)(s^2 + 1)$, we get

$$3s + 1 = A(s^2 + 1) + (Bs + C)(s + 1). \quad ...(1)$$

$s = -1$ in (1) $\Rightarrow$ $-2 = 2A + 0$ $\Rightarrow$ $A = -1$

Comparing coefficients of s^2 and s in (1), we get

$$0 = A + B \quad ...(2) \qquad \text{and} \qquad 3 = B + C \quad ...(3)$$

(2) $\Rightarrow$ B = – A = – (– 1) = 1

(3) $\Rightarrow$ C = 3 – B = 3 – 1 = 2.

$$\therefore \quad \frac{3s+1}{(s+1)(s^2+1)} = \frac{-1}{s+1} + \frac{1.s+2}{s^2+1} = -\frac{1}{s+1} + \frac{s}{s^2+1} + \frac{2}{s^2+1}$$

$$\therefore \quad L^{-1}\left(\frac{3s+1}{(s+1)(s^2+1)}\right) = -L^{-1}\left(\frac{1}{s+1}\right) + L^{-1}\left(\frac{s}{s^2+1}\right) + 2L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= \mathbf{-e^{t} + \cos t + 2\sin t.}$$

(*ii*) Let $$F(s) = \frac{2s^3 + 2s^2 + 4s + 1}{(s^2 + 1)(s^2 + s + 1)}.$$

$\therefore$ F(s) is a proper fraction.

Let $$\frac{2s^3 + 2s^2 + 4s + 1}{(s^2 + 1)(s^2 + s + 1)} = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + s + 1}.$$

Multiplying both sides by $(s^2 + 1)(s^2 + s + 1)$, we get

$$2s^3 + 2s^2 + 4s + 1 = (As + B)(s^2 + s + 1) + (Cs + D)(s^2 + 1).$$

Comparing coefficients of s^3, s^2, s and constant terms, we get

$$2 = A + C \qquad ...(1)$$

$$2 = A + B + D \qquad ...(2)$$

$$4 = A + B + C \qquad ...(3)$$

$$1 = B + D \qquad ...(4)$$

(2) – (4) $\Rightarrow$ $A = 1$

(1) $\Rightarrow$ $C = 2 - A = 2 - 1 = 1$

(3) $\Rightarrow$ $B = 4 - A - C = 4 - 1 - 1 = 2$

(4) $\Rightarrow$ $D = 1 - B = 1 - 2 = -1.$

$$\therefore \qquad F(s) = \frac{1.s + 2}{s^2 + 1} + \frac{1.s - 1}{s^2 + s + 1} = \frac{s}{s^2 + 1} + 2\left(\frac{1}{s^2 + 1}\right) + \frac{s - 1}{s^2 + s + 1}$$

$$\therefore \qquad L^{-1}(F(s)) = L^{-1}\left(\frac{s}{s^2 + 1}\right) + 2L^{-1}\left(\frac{1}{s^2 + 1}\right) + L^{-1}\left(\frac{s - 1}{s^2 + s + 1}\right)$$

$$= \cos t + 2 \sin t + L^{-1}\left(\frac{\left(s + \frac{1}{2}\right) - \frac{3}{2}}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}}\right)$$

$$= \cos t + 2 \sin t + e^{-\frac{1}{2}t} L^{-1}\left(\frac{s - \frac{3}{2}}{s^2 + \frac{3}{4}}\right)$$

$$= \cos t + 2 \sin t + e^{-\frac{1}{2}t} L^{-1}\left(\frac{s}{s^2 + \frac{3}{4}} - \sqrt{3}\,\frac{\sqrt{3}/2}{s^2 + \frac{3}{4}}\right)$$

$$= \mathbf{\cos t + 2 \sin t + e^{-\frac{1}{2}t}\left(\cos \frac{\sqrt{3}}{2} t - \sqrt{3} \sin \frac{\sqrt{3}}{2} t\right)}.$$

Example 6. *Find the inverse Laplace transform of the function* $\dfrac{s}{s^4 + 4a^4}$.

Sol. $$\frac{s}{s^4 + 4a^4} = \frac{s}{(s^2 + 2a^2)^2 - 4a^2s^2} = \frac{s}{(s^2 + 2as + 2a^2)(s^2 - 2as + 2a^2)}$$

$\therefore$ Let $$\frac{s}{s^4 + 4a^4} = \frac{As + B}{s^2 + 2as + 2a^2} + \frac{Cs + D}{s^2 - 2as + 2a^2}.$$

Multiplying both sides by $s^4 + 4a^4$, we get

$$s = (As + B)(s^2 - 2as + 2a^2) + (Cs + D)(s^2 + 2as + 2a^2).$$

Comparing coefficients of s^3, s^2, s and constant terms, we get

$$0 = A + C \qquad ...(1)$$

$$0 = -2aA + B + 2aC + D \qquad ...(2)$$

$$1 = 2a^2A - 2aB + 2a^2C + 2aD \qquad ...(3)$$

$$0 = 2a^2B + 2a^2 D \qquad ...(4)$$

$$(4) \Rightarrow B + D = 0 \qquad ...(5)$$

$$(2) - (5) \Rightarrow -2a(A - C) = 0 \Rightarrow A = C \qquad ...(6)$$

$\therefore$ (1) and (6) $\Rightarrow$ $A = 0, C = 0$

$$(3) \Rightarrow 1 = -2a(B - D) \Rightarrow B - D = -1/2a \qquad ...(7)$$

Solving (5) and (7), we get $B = -1/4a$, $D = 1/4a$.

$$\therefore \quad \frac{s}{s^4 + 4a^4} = \frac{0.s - 1/4a}{s^2 + 2as + 2a^2} + \frac{0.s + 1/4a}{s^2 - 2as + 2a^2}$$

$$= -\frac{1}{4a}\left(\frac{1}{s^2 + 2as + a^2}\right) + \frac{1}{4a}\left(\frac{1}{s^2 - 2as + 2a^2}\right)$$

$$= -\frac{1}{4a}\left(\frac{1}{(s + a)^2 + a^2}\right) + \frac{1}{4a}\left(\frac{1}{(s - a)^2 + a^2}\right)$$

$$\therefore L^{-1}\left(\frac{s}{s^4 + 4a^4}\right) = -\frac{1}{4a}L^{-1}\left(\frac{1}{(s + a)^2 + a^2}\right) + \frac{1}{4a}L^{-1}\left(\frac{1}{(s - a)^2 + a^2}\right)$$

$$= -\frac{1}{4a}e^{-at}L^{-1}\left(\frac{1}{s^2 + a^2}\right) + \frac{1}{4a}e^{at}L^{-1}\left(\frac{1}{s^2 + a^2}\right)$$

$$= \frac{1}{4a}(e^{at} - e^{-at}).\frac{1}{a}L^{-1}\left(\frac{a}{s^2 + a^2}\right) = \frac{1}{2a^2}\left(\frac{e^{at} - e^{-at}}{2}\right).\sin at$$

$$= \mathbf{\frac{1}{2a^2} \sinh at \sin at.}$$