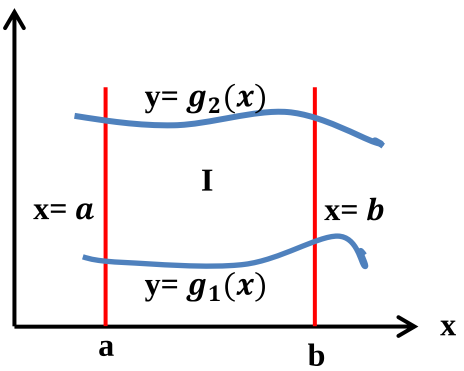
Applied of Double Integration in Calculate Areas and Volumes
تطبيقات التكاملات الثنائية في حساب المساحات والحجوم

Area (المساحة):

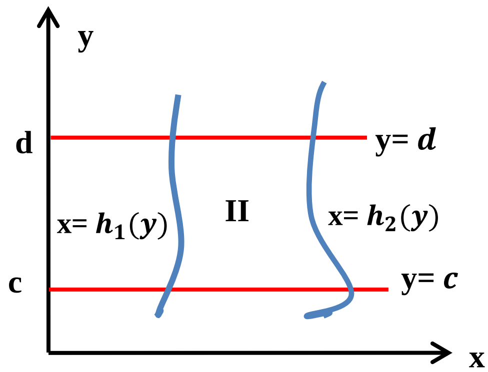
Definition: A type **I** region in bounded on the left and right by vertical lines $x = a$ and $x = b$, and bounded below and above by continuous curve $y = g_1(x)$ & $y = g_2(x)$, where $g_1(x) \leq g_2(x)$, $\forall \;\; a \leq x \leq b$, then

$$A = \iint\limits_{\mathbf{R}} f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dy\, dx$$



Definition: A type **II** region in bounded below and above by horizontal lines $y = c$ and $y = d$, and bounded on the left and right by continuous curve $x = h_1(y)$ & $x = h_2(y)$, where $h_1(y) \leq h_2(y)$, $\forall \;\; c \leq y \leq d$, then

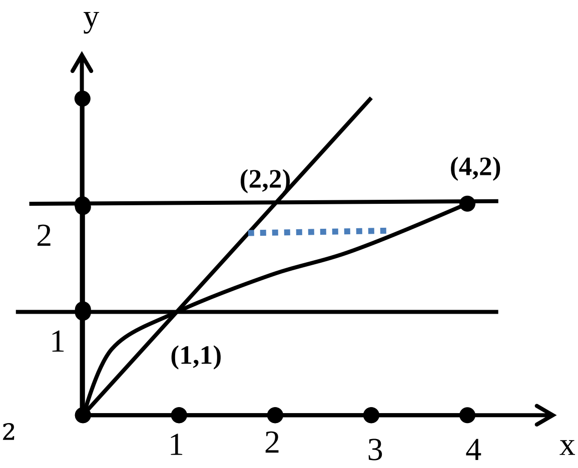
$$A = \iint\limits_{\mathbf{R}} f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dx\, dy$$



where $R = \{(x,y): a \leq x \leq b\,,\;\; c \leq y \leq d\}$

$$\mathrm{dA} = \mathrm{dxdy} = \mathrm{dydx}$$

Note: If there exist the function $f(x,y)$ in double integrals then required to find **volume (الحجم),** if the double integrals without the function $f(x,y)$**,** then required to find **area (المساحة).**



Example 1:

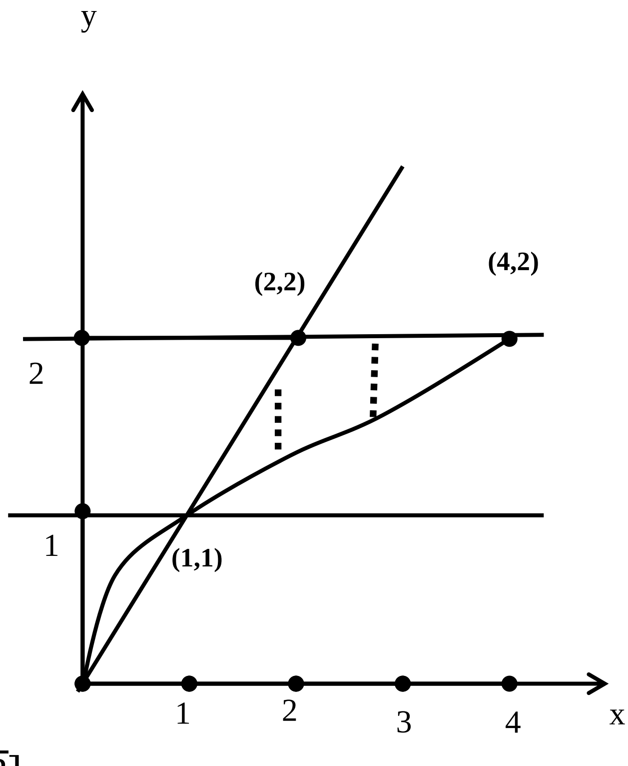
$$\int_1^2\int_y^{y^2} dx\,dy$$

Solution: $y=1,\ y=2$ and $x=y,\ x=y^2$

$$\int_1^2\int_y^{y^2} dx\,dy=\int_1^2 x|_y^{y^2}\,dy=\int_1^2[y^2-y]\,dy=\frac{y^3}{3}-\frac{y^2}{2}\Big|_1^2=\frac{8}{3}-2-\left[\frac{1}{3}-\frac{1}{2}\right]$$

$$=\frac{7}{3}-\frac{3}{2}=\frac{14-9}{6}=\frac{5}{6}$$

Reverse:



$$\int_1^2\int_{\sqrt{x}}^{x} dydx+\int_2^4\int_{\sqrt{x}}^{2} dydx=$$

$$=\int_1^2 y|_{\sqrt{x}}^{x}\,dx+\int_2^4 y|_{\sqrt{x}}^{2}\,dx$$

$$=\int_1^2(x-\sqrt{x})\,dx+\int_2^4(2-\sqrt{x})\,dx$$

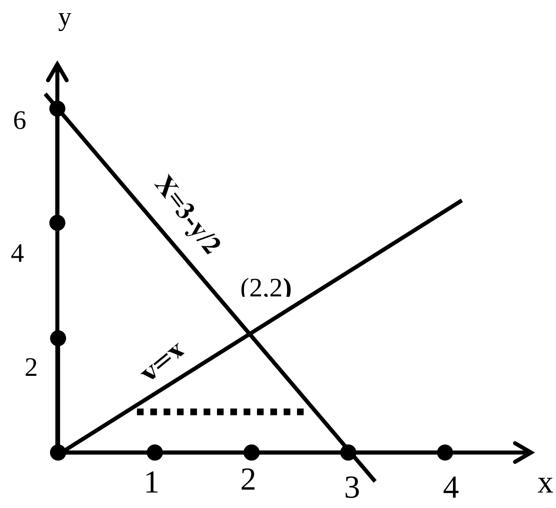
$$=\left(\frac{x^2}{2}-\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)\Bigg|_1^2+\left(2x-\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)\Bigg|_2^4$$

$$=2-\frac{4\sqrt{2}}{3}-\left[\frac{1}{2}-\frac{2}{3}\right]+8-\frac{16}{3}-\left[4-\frac{4\sqrt{2}}{3}\right]$$

$$=2-\frac{1}{2}+\frac{2}{3}+8-\frac{16}{3}-4=\frac{12-3+2+48-32-24}{6}=\frac{5}{6}$$

Example 2:

$$\int_0^2 \int_y^{3-\frac{y}{2}} dx\, dy$$



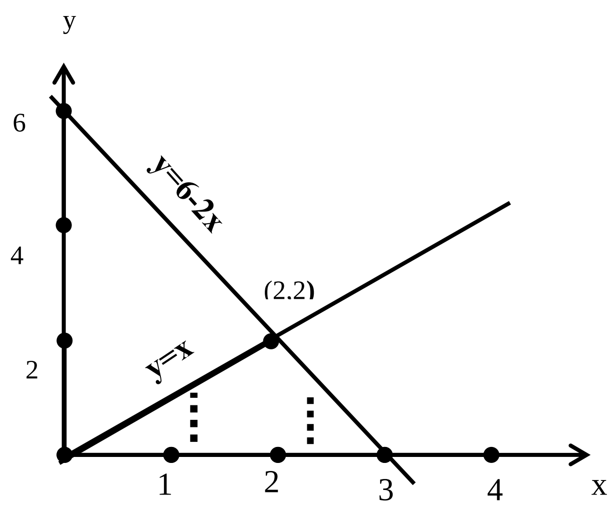
Solution:

$y = 0,\ y = 2\ ,\ x = y,\ x = 3 - \frac{y}{2}$

$$\int_0^2 \int_y^{3-\frac{y}{2}} dx\, dy = \int_0^2 x\Big|_y^{3-\frac{y}{2}} dy$$

$$\int_0^2 \left[3 - \frac{y}{2} - y\right] dy = \int_0^2 \left[3 - \frac{3y}{2}\right] dy = \left[3y - \frac{3y^2}{4}\right]_0^2 = 6 - 3 = 3$$

Reverse**:**



$$\int_0^2 \int_0^x dy dx + \int_2^3 \int_0^{6-2x} dy dx =$$

$$= \int_0^2 y|_0^x\, dx + \int_2^3 y|_0^{6-2x}\ dx$$

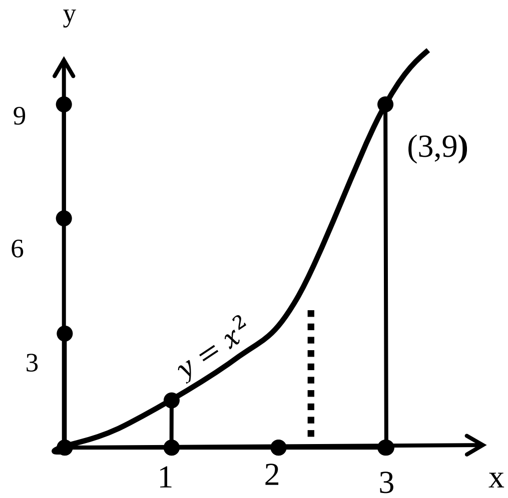
$$= \int_0^2 x\, dx + \int_2^3 (6 - 2x)\ dx$$

$$= \left(\frac{x^2}{2}\right)\Big|_0^2 + (\,6x - x^2)|_2^3$$

$$= 2 + 18 - 9 - (12 - 4) = 3$$

Example 3: Using the double integral to find the area which represented by the curve $y = x^2$ with the x-axis and the line $x = 1, x = 3$. Reversing the order of integral

Solution:



$$\boldsymbol{A} = \int_1^3 \int_0^{x^2} \boldsymbol{dydx} = \int_1^3 \boldsymbol{y}|_0^{x^2} \boldsymbol{dx} = \int_1^3 \boldsymbol{x^2 dx}$$

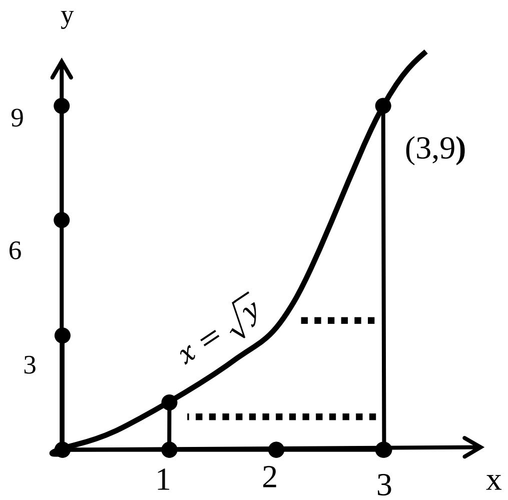
$$== \left[\frac{x^3}{3}\right]_1^3 = \mathbf{9} - \frac{\mathbf{1}}{\mathbf{3}} = \frac{\mathbf{26}}{\mathbf{3}}$$

Reversing:

$$\int_0^1 \int_1^3 \boldsymbol{dxdy} + \int_1^9 \int_{\sqrt{y}}^3 \boldsymbol{dxdy} =$$

$$= \int_0^1 x|_1^3 \, dy + \int_1^9 x|_{\sqrt{y}}^3 \, dy$$

$$= \int_0^1 2 \, dy + \int_1^9 (3 - \sqrt{\boldsymbol{y}}) \, dy$$

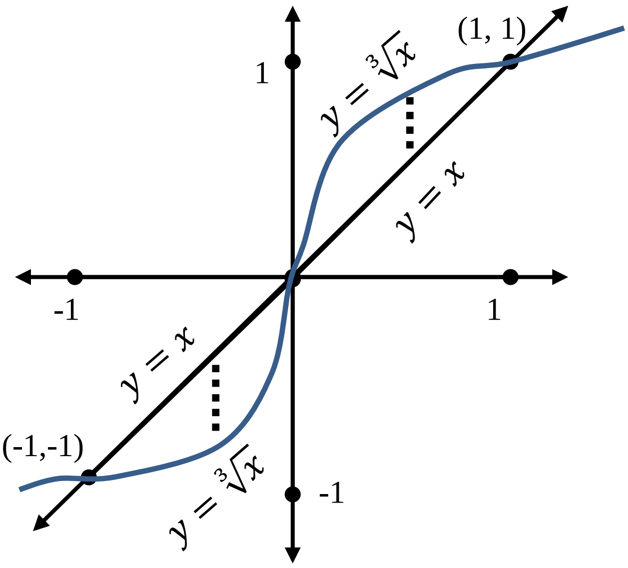


$$= 2y|_{\mathbf{0}}^1 + 3y - \left.\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right|_1^9$$

$$= 2 + \mathbf{27} - \mathbf{18} - \left[\mathbf{3} - \frac{\mathbf{2}}{\mathbf{3}}\right] = \frac{\mathbf{26}}{\mathbf{3}}$$

Example 4: Find the area between the two curves $y = x,\ y = \sqrt[3]{x}$

Solution:



$\boldsymbol{x = \sqrt[3]{x} \quad \Rightarrow \quad x^3 = x \ \Rightarrow \ x^3 - x = 0}$

$\boldsymbol{x(x^2 - 1) = 0}$

$\boldsymbol{x} = 0 \quad \Rightarrow y = 0$

Or $\boldsymbol{x} = 1 \quad \Rightarrow y = 1$

Or $\boldsymbol{x} = -1 \quad \Rightarrow y = -1$

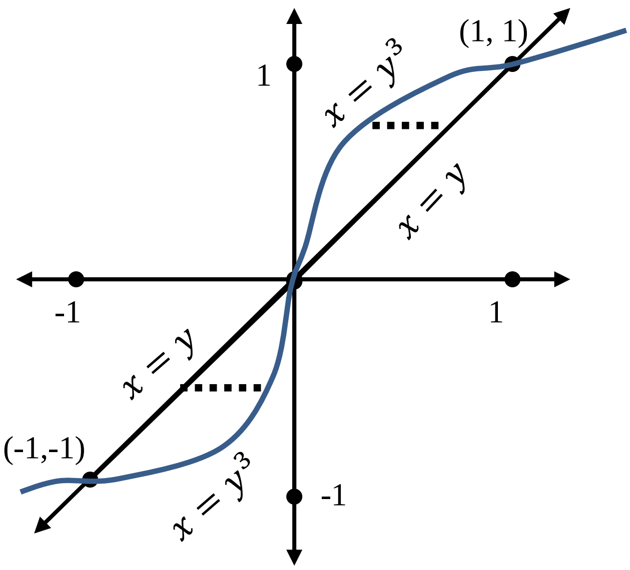
$$\int_{-1}^{0}\int_{\sqrt[3]{x}}^{x} dydx + \int_{0}^{1}\int_{x}^{\sqrt[3]{x}} dydx =$$

$$= \int_{-1}^{0} y|_{\sqrt[3]{x}}^{x}\, dx + \int_{0}^{1} y|_{x}^{\sqrt[3]{x}}\ dx = \int_{-1}^{0} (x - \sqrt[3]{x})\, dx + \int_{0}^{1} (\sqrt[3]{x} - x)\ dx$$

$$= \left(\frac{x^2}{2} - \frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right)\Bigg|_{-1}^{0} + \left(\frac{x^{\frac{4}{3}}}{\frac{4}{3}} - \frac{x^2}{2}\right)\Bigg|_{0}^{1} = -\frac{1}{2} + \frac{3}{4} + \frac{3}{4} - \frac{1}{2} = \frac{1}{2}$$

Reversing:

$$\int_{-1}^{0}\int_{y}^{y^3} dxdy + \int_{0}^{1}\int_{y^3}^{y} dxdy =$$



$$= \int_{-1}^{0} x|_{y}^{y^3}\, dy + \int_{0}^{1} x|_{y^3}^{y}\ dy$$

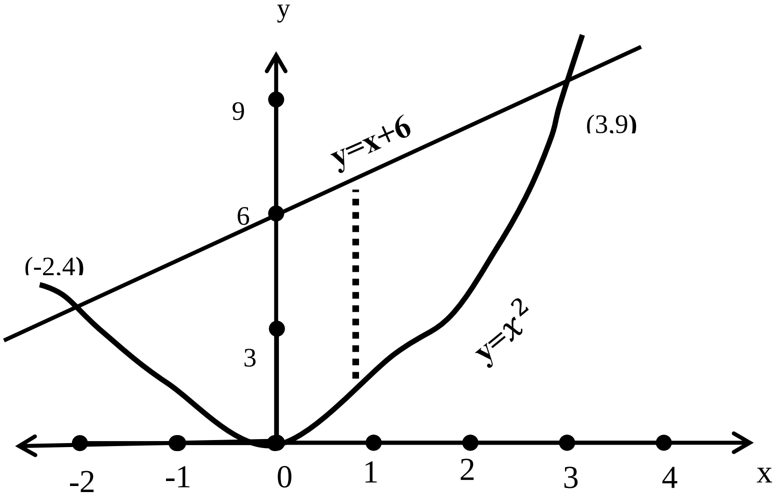
$$= \int_{-1}^{0} (y^3 - y)\, dy + \int_{0}^{1} (y - y^3)\ dy$$

$$= \left(\frac{y^4}{4} - \frac{y^2}{2}\right)\Bigg|_{-1}^{0} + \left(\frac{y^2}{2} - \frac{y^4}{4}\right)\Bigg|_{0}^{1}$$

$$= -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{1}{2}$$

Example 5: Find the area of the region bounded by the curve $y = x^2$, $y = x + 6$

Solution:



$x^2 = x + 6 \quad \Rightarrow \quad x^2 - x - 6 = 0$

$\Rightarrow (\boldsymbol{x} - \mathbf{3})(\boldsymbol{x} + \mathbf{2}) = \mathbf{0}$

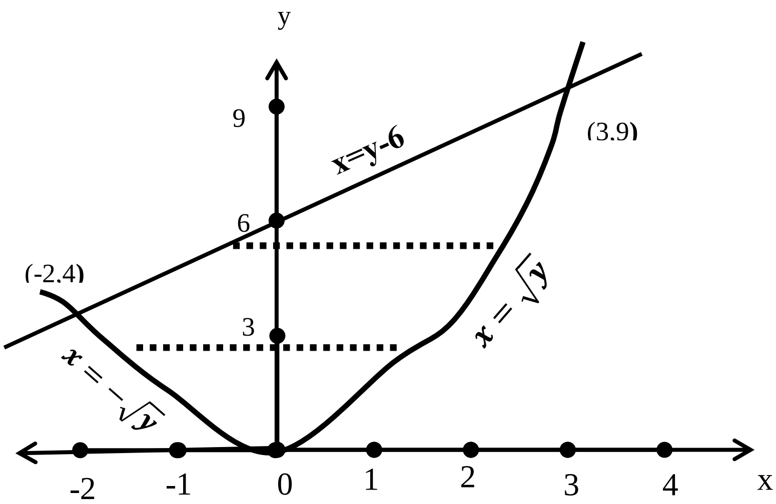
$\boldsymbol{x} = 3 \quad \Rightarrow \ y = 9$

Or $\boldsymbol{x} = -2 \quad \Rightarrow \ y = 4$

$$\int_{-2}^{3}\int_{x^2}^{x+6} \boldsymbol{dydx} = \int_{-2}^{3} y|_{x^2}^{x+6}\, dx$$

$$= \int_{-2}^{3}(x + 6 - x^2)\, dx = \frac{x^2}{2} + 6x - \frac{x^3}{3}\Big|_{-2}^{3} =$$

Reversing:



$$\int_{0}^{4}\int_{-\sqrt{y}}^{\sqrt{y}} \boldsymbol{dxdy} + \int_{4}^{9}\int_{y-6}^{\sqrt{y}} \boldsymbol{dxdy} =$$

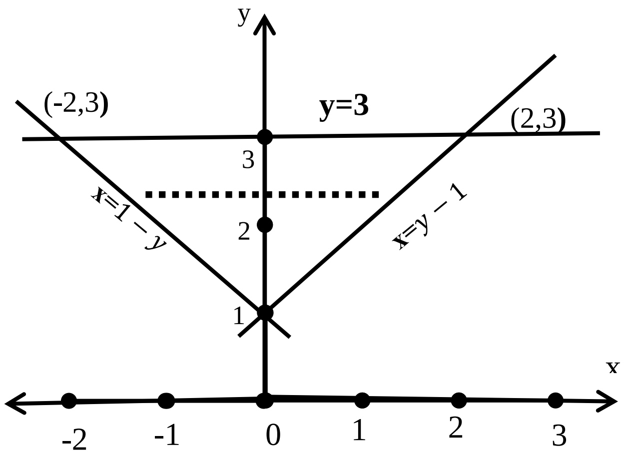
$$= \int_0^4 x|_{-\sqrt{y}}^{\sqrt{y}}\, dy + \int_4^9 x|_{\boldsymbol{y-6}}^{\sqrt{y}}\, dy$$

$$= \int_0^4 (\sqrt{y} + \sqrt{y})\, dy + \int_4^9 (\sqrt{y} - \boldsymbol{y} + \mathbf{6})\, dy$$

$$= \frac{2y^{\frac{3}{2}}}{\frac{\mathbf{3}}{\mathbf{2}}}\Bigg|_0^4 + \left(\frac{y^{\frac{3}{2}}}{\frac{\mathbf{3}}{\mathbf{2}}} - \frac{y^2}{2} + 6y\right)\Bigg|_4^9 =$$

Example 6: Evaluate $\iint_R dA$, over the triangular region enclosed between the lines $y = 1 - x, y = 1 + x, y = 3$, Reversing the order of integration

Solution:



$3 = 1 - x \quad \Rightarrow \quad x = -2$
$\Rightarrow (\mathbf{-2}, \mathbf{3})$

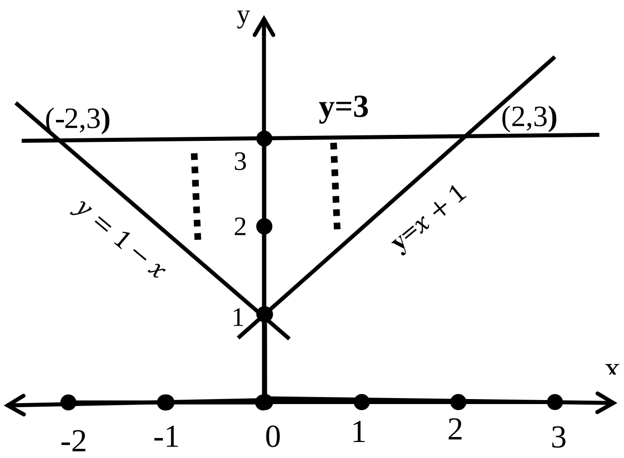
$y = 1 - x, \qquad y = 1 + x \Rightarrow$
$\Rightarrow 1 - x = 1 + x \Rightarrow 2x = 0$
$\Rightarrow x = 0 \Rightarrow y = 1 \Rightarrow (\mathbf{0}, \mathbf{1})$

$3 = 1 + x \quad \Rightarrow \quad x = 2$
$\Rightarrow (\mathbf{2}, \mathbf{3})$

$$\int_{\mathbf{1}}^{\mathbf{3}} \int_{1-y}^{y-1} \boldsymbol{dxdy} = \int_1^3 x|_{1-y}^{y-1} \, dy = \int_1^3 (y - 1 - (1 - y)) \, dy$$

$$\int_1^3 [2y - y] \, dy = y^2 - 2y|_1^3 = 9 - 6 - (1 - 2) = 3 - (-1) = 4$$

Reversing:



$$\int_{\mathbf{-2}}^{\mathbf{0}} \int_{1-x}^{3} \boldsymbol{dydx} + \int_{\mathbf{0}}^{\mathbf{2}} \int_{\boldsymbol{x+1}}^{3} \boldsymbol{dydx} =$$

$$= \int_{-2}^{0} y|_{1-x}^{3} \, dx + \int_0^2 y|_{\boldsymbol{x+1}}^{3} \, dx$$

$$= \int_{-2}^{0} (3 - (1 - x)) \, dx + \int_0^2 (3 - (\boldsymbol{x} + \mathbf{1}) \, dx$$

$$= \int_{-2}^{0} (3 + x) \, dx + \int_0^2 (2 - x) \, dx = 2x + \frac{x^2}{2}\bigg|_{-2}^{0} + 2x - \frac{x^2}{2}\bigg|_0^2$$

$$= -(-4 + 2) + 4 - 2 = 4$$

Exercises:

1. Evaluate $\iint_R \; xy\,dA$ over the triangular R with vertices (رؤوسه) at (0,0), (1,0), (1,1)
2. Find the area between the coordinate axes and the line $x + y = 2$.
3. Find the area between the lines $x = 0,\; y = 2x$ and $y = 4$
4. Evaluate the double integral and reverse the order of integration and evaluate it $\int_{\mathbf{0}}^{\mathbf{1}} \int_{y^2}^{2-y} x dx\, dy$.