9th Lecture

Chapter Three
الـتــرابـــط
Connectedness

الـفـصـل
Definition (3.1): (Separation)

قابلة للفصل
Let (X, τ) be a topological space. We say that the set E is **separable** in (X, τ), if there is two sets A, B such that

(i) $A, B \neq \emptyset$

(ii) $A \cap B = \emptyset$

(iii) $A \cup B = E$

(iv) $[A \cap d(B)] \cup [B \cap d(A)] = \emptyset$

Remark (3.1): From (*iii*) and (*iv*) we get

$[(A \cap \bar{B}) \cup (B \cap \bar{A})] = \emptyset$ ………………….. (*)

شـرط قـابـلـيـة الـفـصـل
The relation (*) is called **separability condition**.

Example (3.1): Let $X = \{a, b, c, d, e\}$ and $\tau = \{\emptyset, \{a, b, c\}, \{c, d, e\}, \{c\}, X\}$.

Determine whether $E = \{a, d, e\}$, $G = \{b, d, e\}$ are separable or not.

Solution:

(1) $E = \{a, d, e\}$

Let $A = \{a, d\}$, $B = \{e\}$

$A, B \neq \emptyset$, $A \cap B = \emptyset$

The family of closed subsets of X are:

X, $\{d, e\}$, $\{a, b\}$, $\{a, b, d, e\}$, $\emptyset$

$\bar{A} = \overline{\{a, d\}} = \bigcap_{F \supset A} F = X \cap \{a, b, d, e\} = \{a, b, d, e\}$

$\bar{B} = \overline{\{e\}} = \bigcap_{F \supset B} F = X \cap \{d, e\} \cap \{a, b, d, e\} = \{d, e\}$

$\Rightarrow \ [(A \cap \bar{B}) \cup (B \cap \bar{A})] = \{d\} \cup \{e\} = \{d, e\} \neq \emptyset$

If $A = \{a\}$, $B = \{d, e\}$

$A, B \neq \emptyset$, $A \cap B = \emptyset$

$\bar{A} = \overline{\{a\}} = \bigcap_{F \supset A} F = X \cap \{a, b\} \cap \{a, b, d, e\} = \{a, b\}$

$\bar{B} = \overline{\{d, e\}} = \bigcap_{F \supset B} F = X \cap \{d, e\} \cap \{a, b, d, e\} = \{d, e\}$

$\Rightarrow \ [(A \cap \bar{B}) \cup (B \cap \bar{A})] = \emptyset \cup \emptyset = \emptyset$

$\Rightarrow$ E is separable in (X, τ).

(2) $G = \{b, d, e\}$

Let $A = \{b\}$, $B = \{d, e\}$

$A, B \neq \emptyset$, $A \cap B = \emptyset$

$\bar{A} = \overline{\{b\}} = \bigcap_{F \supset A} F = X \cap \{a, b\} \cap \{a, b, d, e\} = \{a, b\}$

$\bar{B} = \overline{\{d, e\}} = \bigcap_{F \supset B} F = X \cap \{d, e\} \cap \{a, b, d, e\} = \{d, e\}$

$\Rightarrow \ [(A \cap \bar{B}) \cup (B \cap \bar{A})] = \emptyset \cup \emptyset = \emptyset$

$\Rightarrow$ G is separable in (X, τ).

Note (3.1): The set E is separable in (X, τ) we denote it by $= A/B$.

Definition (3.2): (Connectedness) الترابط

Let (X, τ) be a topological space. We say that a subset E is **connected** (مترابطة) if there does not exist a separation for E in (X, τ).

Example (2.3):

(1) In any topological space (X, τ), the empty set $\emptyset$ and the universal set X are connected.

(2) In any topological space (X, τ), the singleton set $\{x\}$ is always connected.

Theorem (3.1): The (X^*, τ^*) is a topological subspace of (X, τ). Then E is connected in (X, τ) iff E is connected in (X^*, τ^*).

Proof:

Let $E \subset X^* \subset X$

Let $A, B \neq \emptyset$ and $A \cap B = \emptyset$

$\Rightarrow \; A, B \subset E \subset X^*$

Now, if E is connected in (X, τ). Then

$\Leftrightarrow \; (A \cap \bar{B}) \cup (\bar{A} \cap B) \neq \emptyset$

$\Leftrightarrow \; [(A \cap X^*) \cap \bar{B}] \cup [\bar{A} \cap (B \cap X^*)] \neq \emptyset$

$\Leftrightarrow \; [A \cap (X^* \cup \bar{B})] \cup [(X^* \cap \bar{A}) \cap B] \neq \emptyset$

$\Leftrightarrow \; [A \cap \overline{B^*}] \cup [\overline{A^*} \cap B] \neq \emptyset$

$\Leftrightarrow \; E$ is connected in (X^*, τ^*).
