خواص المقدر الجيد

Properties of good Estimator :-

عدم التحيز

1- Unbiasedness :-

*** def:-**

Let $x_1, x_2, \ldots, x_n$ be a random sample from a population having p.d.f. $f(x;\theta)$ with unknown parameter θ. Let $\hat{\theta}$ be an estimator of θ, therefore $\hat{\theta}$ is said to be unbiased estimator of θ if and only if :

$$\boxed{E(\hat{\theta}) = \theta}$$

Otherwise the bias, B, of an estimator $\hat{\theta}$ can be defined as :-

$$\boxed{B = E(\hat{\theta}) - \theta}$$

Generally, we can say that $T(x)$ is an unbiased estimator of $g(\theta)$ if and only if :-

$$E(T(x)) = g(\theta) \quad \forall \theta$$

* **Example :- 1**

Let $x_1, x_2, \dots, x_n$ be a r.s. from $N(\mu, \sigma^2)$.
Show that $\bar{x}$ is an unbiased estimator of μ

solu:

We say that $\bar{x}$ is an unbiased estimator of μ if:

$$E(\bar{x}) = \mu$$

:. نقوم بحساب $E(x)$

$$E(\bar{x}) = E\left[\frac{\sum_{i=1}^{n} x_i}{n}\right] = \frac{\sum_{i=1}^{n} E(x_i)}{n}$$

since $E(x_i) = \mu$ for all $i = 1, 2, \dots, n$, which comes from the property of a random sample that $\bar{x}$ have identical distribution. Then

$$E(\bar{x}) = \frac{n\mu}{n} = \mu$$

Hence $\bar{x}$ is an ubiased estimator of μ.

* **Example :- 2**

Let $Y_1, Y_2, \dots, Y_n$ be a r.s. from $N(\mu, \sigma^2)$ with $E(Y_i) = \mu$ and variance$(Y_i) = \sigma^2$. Show that $S^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$ is a biased estimator of σ^2.

<u>Solu:</u>

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i-\bar{y})^2$$

$$ES^2 = E\left[\frac{1}{n}\sum_{i=1}^{n}(y_i-\bar{y})^2\right] = \frac{1}{n}E\left[\sum_{i=1}^{n}(y_i-\bar{y})^2\right]$$

$$= \frac{1}{n}E\left[\sum_{i=1}^{n}(y_i-\mu-\bar{y}+\mu)^2\right]$$ باضافة وطرح μ

$$= \frac{1}{n}E\left[\sum_{i=1}^{n}\{(y_i-\mu)-(\bar{y}-\mu)\}^2\right]$$

$$= \frac{1}{n}E\left[\sum(y_i-\mu)^2 - 2(\bar{y}-\mu)\sum_{i=1}^{n}(y_i-\mu) + n(\bar{y}-\mu)^2\right]$$

$$= \frac{1}{n}E\left[\sum(y_i-\mu)^2 - 2(\bar{y}-\mu)(n\bar{y}-n\mu) + n(\bar{y}-\mu)^2\right]$$

$$= \frac{1}{n}E\left[\sum_{i=1}^{n}(y_i-\mu)^2 - 2n(\bar{y}-\mu)^2 + n(\bar{y}-\mu)^2\right]$$

$$= \frac{1}{n}E\left[\sum_{i=1}^{n}(y_i-\mu)^2 - n(\bar{y}-\mu)^2\right]$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n}E(y_i-\mu)^2 - nE(\bar{y}-\mu)^2\right]$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n}\mathrm{var}(y_i) - n\,\mathrm{var}(\bar{y})\right]$$

$$= \frac{1}{n}\left[n\sigma^2 - n\frac{\sigma^2}{n}\right]$$

$$= \frac{1}{n}\left[n\sigma^2 - \sigma^2\right]$$

$$= \frac{1}{n}\sigma^2(n-1)$$

$$\therefore ES^2 = \frac{n-1}{n}\sigma^2$$

Hence $S^2 = \frac{1}{n}\sum(y_i-\bar{y})^2$ is a biased estimator of

but $\frac{1}{n-1}\sum_{i=1}^{n}(y_i-\bar{y})^2$ is an unbiased estimator of σ^2.

* **H.W.**

prove that $\frac{1}{n-1}\sum_{i=1}^{n}(y_i-\bar{y})^2$ is an unbiased estimator of σ^2.

* **Example:**

If $ES^2=\frac{n-1}{n}\sigma^2$ where $S^2=\frac{1}{n}\sum_{i=1}^{n}(y_i-\bar{y})^2$,

what is the unbiased estimate for σ^2.

solu:

$\because ES^2=\frac{n-1}{n}\sigma^2 \Longrightarrow \frac{n}{n-1}ES^2=\sigma^2$

$\Longrightarrow \frac{n}{n-1}\left[\frac{n-1}{n}\sigma^2\right]=\sigma^2$

$\therefore \frac{n}{n-1}S^2=\frac{n}{n-1}\cdot\frac{1}{n}\sum_{i=1}^{n}(y_i-\bar{y})^2=\frac{\sum_{i=1}^{n}(y_i-\bar{y})^2}{n-1}$

is an unbiased estimate for σ^2.

* حل Example (2) اعلاه بدون اضافة وطرح M .

$S^2=\frac{1}{n}\sum_{i=1}^{n}(y_i-\bar{y})^2$

$=\frac{1}{n}\sum_{i=1}^{n}\left[y_i^2-2y_i\bar{y}+\bar{y}^2\right]$

$=\frac{1}{n}\left[\sum_{i=1}^{n}y_i^2-2\bar{y}\sum_{i=1}^{n}y_i+n\bar{y}^2\right]$

$=\frac{1}{n}\left[\sum_{i=1}^{n}y_i^2-2n\bar{y}^2+n\bar{y}^2\right]$

$=\frac{1}{n}\left[\sum_{i=1}^{n}y_i^2-n\bar{y}^2\right]$

$$\Rightarrow ES^2 = E\left[\frac{1}{n}\left\{\sum_{i=1}^{n} y_i^2 - n\bar{y}^2\right\}\right]$$
$$= \frac{1}{n}\left[\sum_{i=1}^{n} E y_i^2 - nE\bar{y}^2\right]$$
$$= \frac{1}{n}\left[nEy_i^2 - nE\bar{y}^2\right]$$
$$= Ey_i^2 - E\bar{y}^2$$

$$\because \operatorname{var}(y_i) = Ey_i^2 - (Ey_i)^2$$
$$\sigma^2 = Ey_i^2 - \mu^2$$
$$\Longrightarrow Ey_i^2 = \sigma^2 + \mu^2$$

and $\operatorname{var}(\bar{y}) = E\bar{y}^2 - (E\bar{y})^2$

$$\boxed{\begin{array}{l}\text{if } y_i \sim N(\mu, \sigma^2) \\ \Rightarrow \bar{y} \sim N\left(\mu, \frac{\sigma^2}{n}\right)\end{array}}$$

$$\frac{\sigma^2}{n} = E\bar{y}^2 - \mu^2$$
$$\Rightarrow E\bar{y}^2 = \frac{\sigma^2}{n} + \mu^2$$

$$\Rightarrow ES^2 = \sigma^2 + \mu^2 - \frac{\sigma^2}{n} - \mu^2$$
$$= \sigma^2 - \frac{\sigma^2}{n}$$
$$= \frac{n\sigma^2 - \sigma^2}{n}$$
$$= \frac{(n-1)\sigma^2}{n}$$

$$\therefore ES^2 = \frac{n-1}{n}\sigma^2$$

$\therefore S^2$ is a biased estimator of σ^2.