*Example :-

Show that which of the following estimators is unbiased estimator for $p^2$ (where $p$ is the parameter of binomial distribution).

1- $\left(\frac{X}{n}\right)^2$ 2- $\frac{X(X-1)}{n(n-1)}$

Soln:

1- $E\left(\frac{X}{n}\right)^2 = \frac{1}{n^2} E X^2$

$$= \frac{1}{n^2}\left[var(X) + (EX)^2\right]$$

$$= \frac{1}{n^2}\left[npq + n^2p^2\right]$$

$$= \frac{1}{n^2}\left[np(1-p) + n^2p^2\right]$$

$$= \frac{1}{n^2}\left[np - np^2 + n^2p^2\right]$$

$$= \frac{1}{n}\left[p - p^2 + np^2\right] \neq p^2$$

$\therefore \left(\frac{X}{n}\right)^2$ is a biased estimator for $p^2$

2- $E\left[\frac{X(X-1)}{n(n-1)}\right] = \frac{1}{n(n-1)} E(X^2 - X)$

$$= \frac{1}{n(n-1)}\left[EX^2 - EX\right] = \frac{1}{n(n-1)}\left[npq + n^2p^2 - np\right]$$

$$= \frac{1}{n(n-1)}\left[np(1-p) + n^2p^2 - np\right]$$

$$= \frac{1}{n(n-1)}\left[np - np^2 + n^2p^2 - np\right]$$

$$= \frac{1}{n(n-1)}\left[np^2(n-1)\right]$$

$$= p^2$$

$\therefore \frac{X(X-1)}{n(n-1)}$ is unbiased estimator for $p^2$.

* **Example:**

Let $x_1, x_2, \dots, x_n$ be a r.s. from a distribution having p.d.f. $f(x;\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$ , $0 < x < \infty$ , $\theta > 0$ and zero elsewhere. Show that the sample mean $\bar{X}$ is an unbiased statistic for $\theta$ and has variance $\frac{\theta^2}{n}$

**solu:**

$$f(x;\theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & x > 0 \ , \ \theta > 0 \\ 0 & \text{o.w.} \end{cases}$$

$$E\bar{X} = E\left[\frac{\sum x_i}{n}\right] = \sum_{i=1}^{n} \frac{E x_i}{n}$$

$\because X \sim \exp\left(\frac{1}{\theta}\right) \Longrightarrow E(X) = \theta$ and $var(X) = \theta^2$

$$\Longrightarrow E\bar{X} = \frac{n\theta}{n} = \theta$$

$\therefore \bar{X}$ is unbiased estimator for $\theta$.

$$var(\bar{X}) = var\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) = \frac{\sum_{i=1}^{n} var(x_i)}{n^2} = \frac{n\theta^2}{n^2} = \frac{\theta^2}{n}$$

**H.W.:**

Let $\{X_i\}$, $i = 1, 2, \dots, n$ denote a r.s. from $N(0, \theta)$. Show that $\frac{\sum X_i^2}{n}$ is an unbiased statistic for $\theta$ and has variance $\frac{2\theta^2}{n}$.

## * Mean Square Error (MSe)

**def:**

A useful measure of performance of an estimator $\theta\hat{}$ of the unknown parameter $\theta$ is the mean square error $MSe(\hat{\theta})$ which can be defined as:

$$\boxed{MSe(\hat{\theta}) = E(\hat{\theta}-\theta)^2}$$

Which is also can be decomposed into two parts as follows :-

$$\boxed{MSe(\hat{\theta}) = var(\hat{\theta}) + B^2}$$

where $B$ is the bias value of an estimator $\hat{\theta}$

**Theorem**

$$MSe(\hat{\theta}) = var(\hat{\theta}) + B^2 \quad , \text{where } B = E(\hat{\theta}) - \theta$$

**proof:**

$$MSe(\hat{\theta}) = E(\hat{\theta}-\theta)^2$$

بإضافة وطرح $E(\hat{\theta})$

$$MSe = E\left[(\hat{\theta} - E\hat{\theta}) + (E\hat{\theta} - \theta)\right]^2$$

$$= E(\hat{\theta} - E\hat{\theta})^2 + (E\hat{\theta} - \theta)^2 + 2(E\hat{\theta} - \theta)\underbrace{E(\hat{\theta} - E\hat{\theta})}_{zero}$$

* ملاحظة : التوقع يدخل فقط على $\hat{\theta}$ اما $\theta$ و $E\hat{\theta}$ تعتبر ثوابت وتوقعها يساوي صفر

$$= E(\hat{\theta} - E\hat{\theta})^2 + (E\hat{\theta} - \theta)^2$$

$$= var(\hat{\theta}) + B^2$$

where $B^2 = (bias)^2 = \left[E\hat{\theta} - \theta\right]^2$

In searching for a good estimator, we need to have the mean square error as small as possible.

As a special case if two estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ are ubiased for the parameter $\theta$ with different variances, we would prefer one with small variance.

* *Example* :-

Let $X_1, X_2, \dots, X_n$ be a r.s. from a distribution with p.d.f.

$$f(x;\theta) = \theta^x (1-\theta)^{1-x} I_{\{0,1\}}(x)$$

Use the Mse to compare between the two estimators $\bar{X}$ and $X_1$ for $\theta$.

solu:

$$E\bar{X} = \frac{1}{n} E \sum_{i=1}^{n} X_i$$

$$= \frac{1}{n} \sum_{i=1}^{n} E X_i$$

from the property of a r.s. $X_1, X_2, \dots, X_n$ are independent identically distributed, then

$$E\bar{X} = \frac{n\theta}{n} = \theta$$

and also $E X_1 = \theta$

Therefore. both $\bar{X}$ and $X_1$ are unbiased estimators for $\theta$

$$var(\bar{X}) = var\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) = \frac{n\, var(X_i)}{n^2}$$

$$= \frac{var(X)}{n} = \frac{\theta(1-\theta)}{n}$$

but $var(X_1) = \theta(1-\theta)$

Hence $\bar{X}$ is a better estimator for $\theta$ than $X_1$ because $var(\bar{X})$ is less than $var(X_1)$ for $n > 1$, and also $\bar{X}$ has smaller Mse than $X_1$.

* Example:

let $X_1, X_2, X_3$ be a r.s. of size 3 from $N(\mu, \sigma^2)$

Consider two estimators $\bar{X}$ and $Y = \frac{X_1 + 2X_2 + 3X_3}{6}$

which estimator is better ?

Solu:

$E(\bar{X}) = \mu$

$$E(Y) = E\left[\frac{X_1 + 2X_2 + 3X_3}{6}\right] = \frac{1}{6}\left[\mu + 2\mu + 3\mu\right]$$

$$= \mu$$

Then both estimators $\bar{X}$ and $Y$ are unbiased estimators for $\mu$

$var(\bar{X}) = var\left(\frac{\sum X_i}{n}\right) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$ since the observations are indep. and identically distributed

$$var(Y) = var\left(\frac{X_1 + 2X_2 + 3X_3}{6}\right)$$

$$= \frac{1}{36}\left[\sigma^2 + 4\sigma^2 + 9\sigma^2\right]$$

$$= \frac{7}{18}\sigma^2$$

$\because var(\bar{X}) < var(Y)$

$\therefore \bar{X}$ is a better unbiased estimator for $\mu$.