* **Example**

Let $X_1, X_2, \dots, X_n$ be a r.s. from $N(\theta, \sigma^2)$, use Mse to compare between the following two estimators $\bar{X}$ and $Y = \frac{X_1 + 2X_2}{3}$ (which is preferable)

Solu:

$E\bar{X} = \theta$ (because $\bar{X}$ is unbiased estimator for mean of Normal distribution).

$$EY = E\,\frac{X_1 + 2X_2}{3} = \frac{EX_1 + 2EX_2}{3} = \frac{\theta + 2\theta}{3} = \frac{3\theta}{3} = \theta$$

$\Rightarrow \bar{X}$ and Y are unbiased estimators for θ

$$Var(\bar{X}) = \frac{\sigma^2}{n}$$

$$Var(Y) = Var\left(\frac{X_1 + 2X_2}{3}\right) = \frac{Var(X_1) + 4\,Var(X_2)}{9} = \frac{5\sigma^2}{9}$$

$$Mse(\bar{X}) = var(\bar{X}) = \frac{\sigma^2}{n}$$

$$Mse(Y) = var(Y) = \frac{5\sigma^2}{9}$$

$\therefore Mse(\bar{X}) < Mse(Y)$ or $var(\bar{X}) < var(Y)$

$\therefore \bar{X}$ is better estimator than Y.

* **Example:**

Let $X_1, X_2, \dots, X_n$ be a r.s. from binomial distribution then use Mse to compare between the following estimators for θ.

$$Y_1 = \frac{X + \frac{\sqrt{n}}{2}}{n + \sqrt{n}} \quad \text{and} \quad Y_2 = \frac{X_1 + X_2}{2}$$

solu:

$$E\,Y_1 = E\,\frac{X + \frac{\sqrt{n}}{2}}{n+\sqrt{n}} = \frac{E(X) + \frac{\sqrt{n}}{2}}{n+\sqrt{n}} = \frac{n\theta + \frac{\sqrt{n}}{2}}{n+\sqrt{n}} \neq \theta$$

$\Rightarrow Y_1$ is a biased estimator for θ.

$$E(Y_2) = E\,\frac{X_1 + X_2}{2} = \frac{EX_1 + EX_2}{2} = \frac{n\theta + n\theta}{2} = n\theta$$

$\Rightarrow Y_2$ is a biased estimator for θ.

$$MSe(\hat{\theta}) = Var(\hat{\theta}) + b^2(\hat{\theta})$$ بشكل عام

$$\Rightarrow var(Y_1) = Var\left(\frac{X + \frac{\sqrt{n}}{2}}{n+\sqrt{n}}\right) = \frac{Var(X)}{(n+\sqrt{n})^2} = \frac{n\theta(1-\theta)}{(n+\sqrt{n})^2}$$

$$b^2(Y_1) = [E(Y_1) - \theta]^2 = \left[\frac{n\theta + \frac{\sqrt{n}}{2}}{n+\sqrt{n}} - \theta\right]^2$$

$$= \left[\frac{n\theta + \frac{\sqrt{n}}{2} - n\theta - \sqrt{n}\,\theta}{n+\sqrt{n}}\right]^2 = \left[\frac{\sqrt{n}\,(\sqrt{n}\,\theta + \frac{1}{2} - \sqrt{n}\,\theta - \theta)}{\sqrt{n}\,(\sqrt{n}+1)}\right]^2$$

$$= \left(\frac{\frac{1}{2} - \theta}{\sqrt{n}+1}\right)^2$$

$$MSe(Y_1) = Var(Y_1) + b^2(Y_1)$$

$$= \frac{n\theta(1-\theta)}{(n+\sqrt{n})^2} + \frac{(\frac{1}{2}-\theta)^2}{(\sqrt{n}+1)^2} = \frac{\theta(1-\theta)}{\frac{1}{n}(n+\sqrt{n})^2} + \frac{\frac{1}{4} - \theta + \theta^2}{(\sqrt{n}+1)^2}$$

$$= \frac{\theta(1-\theta)}{\left[\frac{1}{\sqrt{n}}(n+\sqrt{n})\right]^2} + \frac{\frac{1}{4} - \theta + \theta^2}{(\sqrt{n}+1)^2} = \frac{\theta(1-\theta)}{(\sqrt{n}+1)^2} + \frac{\frac{1}{4} - \theta + \theta^2}{(\sqrt{n}+1)^2}$$

$$= \frac{\theta - \theta^2 + \frac{1}{4} - \theta + \theta^2}{(\sqrt{n}+1)^2} = \frac{1}{4(\sqrt{n}+1)^2}$$

$$Var(Y_2) = var\left(\frac{X_1+X_2}{2}\right) = \frac{var(X_1)+var(X_2)}{4} = \frac{2n\theta(1-\theta)}{4}$$

$$= \frac{n\theta(1-\theta)}{2}$$

$$b^2(Y_2) = [EY_2 - \theta]^2 = [n\theta - \theta]^2 = \theta^2(n-1)^2$$

$$MSe(Y_2) = Var(Y_2) + b^2(Y_2)$$

$$= \frac{n\theta(1-\theta)}{2} + \theta^2(n-1)^2$$

بما أن $n>1$ وأن $MSe(Y_1)$ متغير n بالمقام و $MSe(Y_2)$ متغير n بالبسط

$$\therefore MSe(Y_1) < MSe(Y_2)$$

$\therefore$ Y_1 is the best estimator.

* H.W:

Let $X_1, X_2, \dots, X_n$ be n-trials of Bernoulli-dist. Then use Mse to show which of the following estimators is better

① $\bar{X} = \frac{1}{n}\sum x_i$ and X_1

② $Y_1 = \frac{X_1+X_2}{2}$ and $Y_2 = X_1$

* Example:

Find MSe for X which is drawn from uniform distribution $(0,\theta)$, i.e $X \sim U[0,\theta]$, where X is an estimator for θ.

Soln:

$$f(x) = \frac{1}{b-a} = \frac{1}{\theta}$$

$$E(x) = \frac{a+b}{2} = \frac{\theta+0}{2} = \frac{\theta}{2}$$

$$b^2(\hat{\theta}) = [E\hat{\theta} - \theta]^2 = [EX - \theta]^2 = [\frac{\theta}{2} - \theta]^2$$

$$= [\frac{\theta - 2\theta}{2}]^2 = \frac{\theta^2}{4}$$

$$Var(\hat{\theta}) = \frac{(b-a)^2}{12} = \frac{[\theta - 0]^2}{12} = \frac{\theta^2}{12}$$

$$Mse(\hat{\theta}) = Mse(x) = \frac{\theta^2}{12} + \frac{\theta^2}{4} = \frac{\theta^2 + 3\theta^2}{12} = \frac{4\theta^2}{12} = \frac{\theta^2}{3}$$