Consistency : الاتساق

Def:

Let $X_1, X_2, ---, X_n$ be a random sample from a population with p.d.f. $f(X;\theta)$, $\theta \in \Omega$. The estimator $\hat{\theta}_n$ is called a consistent estimator for the parameter θ if and only if :-

$$\lim_{n\to\infty} E[\hat{\theta}_n - \theta] = 0 \quad \forall\ \theta \in \Omega$$

Which implies two conditions :-

1- $\lim_{n\to\infty} E(\hat{\theta}_n) = \theta$

2- $\lim_{n\to\infty} Var(\hat{\theta}_n) = 0$

*** Example:**

Let $X_1, X_2, ---, X_n$ be a r.s. from a normal distribution with (μ, σ^2). Are $\bar{X}$ and $S^2 = \frac{1}{n}\Sigma(X_i - \bar{X})^2$ consistent estimators for μ and σ^2 respectively?

Soln:

To check consistency for $\bar{X}$, we do have the following :-

1) $\lim_{n\to\infty} E\bar{X} = \lim_{n\to\infty} \mu = \mu$

2) $\lim_{n\to\infty} var(\bar{X}) = \lim_{n\to\infty} \frac{\sigma^2}{n} = 0$

Hence $\bar{X}$ is a consistent estimator for μ.

Now, for the consistency of S^2.

1) $\lim\limits_{n\to\infty} E S^2 = \lim\limits_{n\to\infty} E\left[\frac{1}{n}\Sigma(X_i-\bar{X})^2\right] = \lim\limits_{n\to\infty}\left(\frac{n-1}{n}\sigma^2\right)$

برهنا سابقا:

بقسمة البسط والمقام على n

$= \lim\limits_{n\to\infty}\left(\frac{\frac{n}{n}-\frac{1}{n}}{\frac{n}{n}}\sigma^2\right) = \lim\limits_{n\to\infty}\left(\frac{1-\frac{1}{n}}{1}\sigma^2\right) = \lim\limits_{n\to\infty}\left(1-\frac{1}{n}\right)\sigma^2$

$= \sigma^2$

2) $\lim\limits_{n\to\infty} var(S^2) = \lim\limits_{n\to\infty}\left[\frac{2(n-1)}{n^2}\right]\sigma^4$

برهنا سابقا:

$= \lim\limits_{n\to\infty}\left[\frac{2}{n}-\frac{2}{n^2}\right]\sigma^4$

$= 0$

Hence $S^2 = \frac{1}{n}\Sigma(X_i-\bar{X})^2$ is a consistent estimator for σ^2.

* Example:

1- Is $\frac{X+1}{n+2}$ a consistent estimator for parameter of binomial distribution ?

2- Is $\bar{X}$ is a consistent estimator for parameter of poisson distribution ?

3- Is $(\bar{X}+1)$ is a consistent estimator for mean of $N(\theta, \sigma^2)$?

solu (1)

$$E\left[\frac{X+1}{n+2}\right] = \frac{EX+1}{n+2} = \frac{n\theta+1}{n+2}$$

1- $$\lim_{n\to\infty} E\left[\frac{X+1}{n+2}\right] = \lim_{n\to\infty}\left[\frac{n\theta+1}{n+2}\right] = \lim_{n\to\infty} \frac{\theta+\frac{1}{n}}{1+\frac{2}{n}} = \theta$$

$\therefore \frac{X+1}{n+2}$ is unbiased estimator for θ in limit

2- $$\lim_{n\to\infty} Var\left[\frac{X+1}{n+2}\right] = \lim_{n\to\infty} \frac{Var(X)+0}{(n+2)^2} = \lim_{n\to\infty} \frac{n\theta(1-\theta)}{n^2+4n+4}$$

بقسمة البسط والمقام على n^2

$$= \lim_{n\to\infty} \frac{\frac{\theta(1-\theta)}{n}}{1+\frac{4}{n}+\frac{4}{n^2}} = 0$$

$\therefore$ from (1) and (2) $\frac{X+1}{n+2}$ is consistent estimator for the parameter of binomial distribution.

solu (2)

① $$\lim_{n\to\infty} E\bar{X} = \lim_{n\to\infty} E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} EX_i = \lim_{n\to\infty} \frac{n\lambda}{n} = \lambda$$

② $$\lim_{n\to\infty} var(\bar{X}) = \lim_{n\to\infty} Var\left[\frac{1}{n}\Sigma X_i\right] = \lim_{n\to\infty}\left[\frac{1}{n^2}\Sigma var(X_i)\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n^2} n\lambda\right] = \lim_{n\to\infty} \frac{\lambda}{n} = 0$$

$\therefore \bar{X}$ is a consistent estimator for parameter of poisson distribution.

solu (3)

① $$\lim_{n\to\infty} E(\bar{X}+1) = \lim_{n\to\infty} E\left[\frac{\Sigma X_i}{n}+1\right] = \lim_{n\to\infty}\left[\frac{\sum_{i=1}^{n} EX_i}{n}+1\right]$$

$$= \lim_{n\to\infty}\left[\frac{n\theta}{n}+1\right] = \lim_{n\to\infty}[\theta+1] = \theta+1$$

$\therefore (\bar{X}+1)$ is non-consistent estimator for the mean of $N(\theta, \sigma^2)$

* **Example:**

Is $\bar{X}$ a consistent estimator for θ of :-

1) If $X_1, \dots, X_n \sim Ber(\theta)$
2) If $X_1, \dots, X_n \sim bin(n, \theta)$

Solu:

① $f(x;\theta) = \theta^x (1-\theta)^{1-x}$

1) $\lim\limits_{n\to\infty} E(\bar{x}) = \lim\limits_{n\to\infty} E\left[\frac{\sum x_i}{n}\right] = \lim\limits_{n\to\infty} \frac{\sum_{i=1}^{n} E x_i}{n} = \lim\limits_{n\to\infty} \frac{\sum_{i=1}^{n}\theta}{n}$

$= \lim\limits_{n\to\infty} \frac{n\theta}{n} = \lim\limits_{n\to\infty} \theta = \theta$

2) $\lim\limits_{n\to\infty} var(\bar{X}) = \lim\limits_{n\to\infty} var\left(\frac{\sum x_i}{n}\right) = \lim\limits_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} var(x_i)$

$= \lim\limits_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} [\theta(1-\theta)] = \lim\limits_{n\to\infty} \frac{n\theta(1-\theta)}{n^2}$

$= \lim\limits_{n\to\infty} \frac{\theta(1-\theta)}{n} = 0$

$\therefore \bar{X}$ is a consistent estimator for θ (the parameter of Bernoulli distribution).

② $f(x;n,\theta) = C_x^n \, \theta^x (1-\theta)^{n-x} \qquad x = 0, 1, 2, \dots, n$

1- $\lim\limits_{n\to\infty} E\bar{X} = \lim\limits_{n\to\infty} E\left(\frac{X}{n}\right) \quad ; \; X = \sum_{i=1}^{n} X_i$ = no. of successes

$\bar{X}$ = proportion of successes

$\lim\limits_{n\to\infty} E\bar{X} = \lim\limits_{n\to\infty} \frac{EX}{n} = \lim\limits_{n\to\infty} \frac{n\theta}{n} = \lim\limits_{n\to\infty} \theta = \theta$

2- $\lim\limits_{n\to\infty} var(\bar{X}) = \lim\limits_{n\to\infty} Var\left(\frac{X}{n}\right) = \lim\limits_{n\to\infty} \frac{var(X)}{n^2}$

$= \lim\limits_{n\to\infty} \frac{n\theta(1-\theta)}{n^2} = \lim\limits_{n\to\infty} \frac{\theta(1-\theta)}{n} = 0$

$\therefore \bar{X}$ is a consistent estimator for θ (the parameter of binomial distribution).