Fisher Information : معلومات فيشر

* **Def:**

The quantity $(n\, I_X(\theta))$ is known as the amount كمية of information about the parameter θ in a r.s. of size n, where

$$I_X(\theta) = E\left[\frac{\partial \ln f(x;\theta)}{\partial\theta}\right]^2$$

$$= -E\left[\frac{\partial^2 \ln f(x;\theta)}{\partial\theta^2}\right] = \text{Fisher information in a random observation } X \text{ about the parameter } \theta.$$

* **Example:-**

Compute fisher information of the parameter of Bernoulli distribution, using a r.s. of size n.

Solu:

$$f(x;\theta) = \theta^x(1-\theta)^{1-x} \qquad x=0,1 \qquad 0<\theta<1$$

$$\ln f(x;\theta) = \ln\left[\theta^x(1-\theta)^{1-x}\right]$$

$$= x\ln\theta + (1-x)\ln(1-\theta)$$

$$\frac{\partial \ln f(x;\theta)}{\partial\theta} = x\,\frac{1}{\theta} + (1-x)\,\frac{(-1)}{1-\theta} \quad ①$$

$$= \frac{x}{\theta} - \frac{1-x}{1-\theta} = \frac{x(1-\theta)-\theta(1-x)}{\theta(1-\theta)}$$

$$= \frac{x-\cancel{x\theta}-\theta+\cancel{x\theta}}{\theta(1-\theta)}$$

$$= \frac{x-\theta}{\theta(1-\theta)}$$

$$\left[\frac{\partial \ln f(x;\theta)}{\partial\theta}\right]^2 = \left[\frac{X-\theta}{\theta(1-\theta)}\right]^2 = \frac{X^2-2X\theta+\theta^2}{\theta^2(1-\theta)^2}$$

$$\therefore I_X(\theta) = E\left[\frac{\partial \ln f(x;\theta)}{\partial\theta}\right]^2 = E\left[\frac{X^2-2X\theta+\theta^2}{\theta^2(1-\theta)^2}\right]$$

$$= \frac{1}{\theta^2(1-\theta)^2}\left[EX^2-2\theta EX+\theta^2\right]$$

$\because EX=\theta$ and $var(X)=\theta(1-\theta)$

$$\because var(X) = EX^2-(EX)^2 \Longrightarrow EX^2 = var(X)+(EX)^2$$
$$= \theta(1-\theta)+\theta^2$$
$$= \theta-\theta^2+\theta^2$$
$$= \theta$$

$$\therefore I_X(\theta) = \frac{1}{\theta^2(1-\theta)^2}\left[\theta-2\theta^2+\theta^2\right] = \frac{\theta-\theta^2}{\theta^2(1-\theta)^2} = \frac{\theta(1-\theta)}{\theta^2(1-\theta)^2}$$

$$= \frac{1}{\theta(1-\theta)}$$

$$\therefore \text{Fisher information} = n\,I_X(\theta) = \frac{n}{\theta(1-\theta)}$$

ويمكن حل نفس المثال باستخدام القانون الثاني لمعلومات فيشر اي بإيجاد المشتقة الثانية وكالتالي :-

من المعادلة ①

$$\frac{\partial^2 \ln f(x;\theta)}{\partial\theta^2} = \frac{-x}{\theta^2} - \frac{(1-x)}{(1-\theta)^2}$$

$$I_X(\theta) = -E\left[\frac{\partial^2 \ln f(x;\theta)}{\partial\theta^2}\right] = -E\left[\frac{-x}{\theta^2} - \frac{1-x}{(1-\theta)^2}\right]$$

$$= \frac{EX}{\theta^2} + \frac{E(1-X)}{(1-\theta)^2} = \frac{\theta}{\theta^2} + \frac{1-\theta}{(1-\theta)^2} = \frac{1}{\theta}+\frac{1}{1-\theta}$$

$$= \frac{1-\theta+\theta}{\theta(1-\theta)} = \frac{1}{\theta(1-\theta)}$$

$$\Longrightarrow \text{Fisher Information} = n\,I_X(\theta) = \frac{n}{\theta(1-\theta)}$$

* **Example:**

Find Fisher information of θ from $N(\theta, \sigma^2)$ using n-observations

solu:

$$f(x;\theta) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x-\theta)^2}$$

$$\ln f(x;\theta) = \ln \frac{1}{\sqrt{2\pi}\,\sigma} - \frac{1}{2\sigma^2}(x-\theta)^2$$

$$\frac{\partial \ln f(x;\theta)}{\partial \theta} = \frac{-2(x-\theta)(-1)}{2\sigma^2} = \frac{x-\theta}{\sigma^2} = \frac{x}{\sigma^2} - \frac{\theta}{\sigma^2}$$

$$\frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2} = -\frac{1}{\sigma^2}$$

$$I_x(\theta) = -E\left[\frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2}\right] = -E\left[\frac{-1}{\sigma^2}\right] = \frac{1}{\sigma^2}$$

The Fisher information in a r.s. of size n is :

$$\text{Fisher information} = n\, I_x(\theta) = \frac{n}{\sigma^2}$$

* **Example:**

Find Fisher Information of the parameter θ from poisson (θ) using n-observations :-

solu:

$$f(x;\theta) = \frac{e^{-\theta}\,\theta^{x}}{x!} \quad , x = 0, 1, 2, \dots$$

$$\ln f(x;\theta) = -\theta + x\ln\theta - \ln x!$$

$$\frac{\partial \ln f(x;\theta)}{\partial \theta} = -1 + \frac{x}{\theta} \implies \frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2} = \frac{-x}{\theta^2}$$

$$\therefore I_x(\theta) = -E\left[\frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2}\right] = -E\left[\frac{-x}{\theta^2}\right] = \frac{Ex}{\theta^2} = \frac{\theta}{\theta^2} = \frac{1}{\theta}$$

$$\therefore \text{Fisher Information} = n\, I_x(\theta) = \frac{n}{\theta}$$

1. Example:

Find Fisher information of σ^2 from $N(\theta,\sigma^2)$ using n-observations

Solu:

$$f(x;\theta,\sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x-\theta)^2}$$

$$\ln f(x;\theta,\sigma^2) = \ln\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right) - \frac{1}{2\sigma^2}(x-\theta)^2$$

$$= -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}(x-\theta)^2$$

$$\frac{\partial \ln f(x;\theta,\sigma^2)}{\partial\sigma^2} = 0 - \frac{1}{2\sigma^2} + \frac{1}{2\sigma^4}(x-\theta)^2$$

$$\frac{\partial^2 \ln f(x;\theta,\sigma^2)}{\partial(\sigma^2)^2} = \frac{1}{2\sigma^4} + \frac{-2(\sigma^2)^{-3}}{2}(x-\theta)^2$$

$$= \frac{1}{2\sigma^4} - \frac{(x-\theta)^2}{\sigma^6}$$

$$I_x(\theta) = -E\left[\frac{\partial^2 \ln f(x;\theta,\sigma^2)}{\partial(\sigma^2)^2}\right] = -\frac{1}{2\sigma^4} + \frac{E(x-\theta)^2}{\sigma^6}$$

$$= \frac{-1}{2\sigma^4} + \frac{Var(x)}{\sigma^6} = \frac{-1}{2\sigma^4} + \frac{\sigma^2}{\sigma^6} = \frac{-1}{2\sigma^4} + \frac{1}{\sigma^4}$$

$$= \frac{-1+2}{2\sigma^4} = \frac{1}{2\sigma^4}$$

$\therefore$ Fisher information $= n\,I_x(\theta) = \dfrac{n}{2\sigma^4}$

Cramer – Rao inequality :- C.R inequality

Let $x_1, x_2, \dots, x_n$ be a r.s. of size n from $f(x;\theta)$ and let $T = t(x_1, x_2, \dots, x_n)$ be unbiased estimator for $g(\theta)$. Then

$$Var(T) \geq \frac{[g'(\theta)]^2}{n\, I_x(\theta)}$$

Cramer – Rao lower bound (C.R.L.B)

The lower bound for the variance of unbiased estimator T is

$$var(T) = \frac{[g'(\theta)]^2}{n\, I_x(\theta)}$$

Remark :

If $g(\theta) = \theta$ Then $E(T) = \theta$ and $g'(\theta) = 1$

$$\Rightarrow var(T) = \frac{1}{n I_x(\theta)}$$

* Efficiency :- الكفاءة

* **Def:**

If we have two estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ for the parameter θ from $f(x;\theta)$, then we say that $\hat{\theta}_1$ is more efficient than $\hat{\theta}_2$ if :-

$$Mse(\hat{\theta}_1) < Mse(\hat{\theta}_2)$$

or $var(\hat{\theta}_1) < var(\hat{\theta}_2)$ if $\hat{\theta}_1$ and $\hat{\theta}_2$ are both ubiased estimators for the parameter θ.

* Relative efficiency : (R.eff)

The relative efficiency of $\hat{\theta}_1$ to $\hat{\theta}_2$ may be defined as:-

$$\text{Relative eff.}\,(\hat{\theta}_1, \hat{\theta}_2) = \frac{MSe(\hat{\theta}_1)}{MSe(\hat{\theta}_2)}$$

$$= \frac{Var(\hat{\theta}_1)}{Var(\hat{\theta}_2)}$$ if $\hat{\theta}_1$ and $\hat{\theta}_2$ are both unbiased estimators for θ.

Therefore, we say that $\hat{\theta}_1$ is relatively efficient than $\hat{\theta}_2$ if $R.eff\,(\hat{\theta}_1, \hat{\theta}_2) < 1$

* Def:

The efficiency of an unbiased estimator $T(x)$ is defined by the ratio

$$eff(T) = \frac{1/n\, I_x(\theta)}{Var(T)}$$

ملاحظة:- بالنسبة لـ $eff(T)$
البسط يمثل $C.R.L.B$
والمقام يمثل تباين المقدر غير المتحيز

The statistic $T(x)$ is called an efficient statistic iff $eff(T) = 1$

* Example:

Let $x_1, x_2, \dots, x_n$ be a r.s. from a poisson distribution with parameter θ. Show that $\bar{x}$ is an efficient statistic for θ.

Solu:

$$f(x;\theta) = \frac{e^{-\theta}\,\theta^{x}}{x!} \qquad x = 0, 1, 2, 3, \dots$$

$$\ln f(x;\theta) = -\theta + x\ln\theta - \ln x!$$

$$\frac{\partial \ln f(x;\theta)}{\partial \theta} = -1 + \frac{x}{\theta}$$

$$E\left[\frac{\partial \ln f(x;\theta)}{\partial\theta}\right]^2 = E\left[\frac{X-\theta}{\theta}\right]^2 = \frac{E(X-\theta)^2}{\theta^2} = \frac{var(X)}{\theta^2}$$

$$= \frac{\theta}{\theta^2} = \frac{1}{\theta} \longrightarrow n I_x(\theta) = \frac{n}{\theta}$$

Hence the C.R. lower bound is $\frac{1}{n I_x(\theta)} = \frac{1}{\frac{n}{\theta}} = \frac{\theta}{n}$

and also it can be found that $var(\bar{X}) = var\left(\frac{\sum X_i}{n}\right)$

$$\Rightarrow var(\bar{X}) = \frac{n\theta}{n^2} = \frac{\theta}{n}$$

There fore $eff(\bar{X}) = \frac{1/n I_x(\theta)}{var(\bar{X})} = \frac{\frac{\theta}{n}}{\frac{\theta}{n}} = 1$

$\therefore$ $\bar{X}$ is an efficient statistic for θ.