* **Example:**

Let $X_1, X_2, \dots, X_n$ be a r.s. of size $n$ from Bernoulli distribution with parameter $\theta$, and let $\bar{X}$ be an estimator for $\theta$. Apply Cramer-Rao inequality on the $var(\bar{X})$, and find the lower bound (L.B.) for $var(\bar{X})$

**Solu:**

$$f(x;\theta) = \theta^{x}(1-\theta)^{1-x} \qquad x = 0, 1$$

$$\ln f(x;\theta) = x \ln\theta + (1-x)\ln(1-\theta)$$

$$\frac{\partial \ln f(x;\theta)}{\partial\theta} = \frac{x}{\theta} - \frac{1-x}{1-\theta}$$

$$\frac{\partial^2 \ln f(x;\theta)}{\partial\theta^2} = \frac{-x}{\theta^2} - \frac{1-x}{(1-\theta)^2}$$

$$I_x(\theta) = -E\left[\frac{\partial^2 \ln f(x;\theta)}{\partial\theta^2}\right] = \frac{EX}{\theta^2} + \frac{E(1-X)}{(1-\theta)^2}$$

$$= \frac{\theta}{\theta^2} + \frac{1-\theta}{(1-\theta)^2} = \frac{1}{\theta} + \frac{1}{1-\theta} = \frac{1-\theta+\theta}{\theta(1-\theta)} = \frac{1}{\theta(1-\theta)}$$

$\longrightarrow$ Fisher Information $= n\, I_x(\theta) = \dfrac{n}{\theta(1-\theta)}$

$$E(\bar{X}) = E\left[\frac{\sum X_i}{n}\right] = \frac{1}{n}\sum_{i=1}^{n} E X_i = \frac{1}{n}\, n\theta = \theta$$

$\therefore$ $\bar{X}$ is unbiased estimator for $\theta$.

We have Cramer-Rao inequality

$$Var(T) \geq \frac{[g'(\theta)]^2}{n\, I_x(\theta)} \implies Var(\bar{X}) \geq \frac{1}{n\, I_x(\theta)}$$

$$\Rightarrow var(\bar{x}) \geq \frac{\theta(1-\theta)}{n}$$

$\therefore$ The lower bound for $var(\bar{X})$ is

$$var(\bar{x}) = \frac{\theta(1-\theta)}{n}$$

Note:

If $\hat{\theta}$ is the unbiased estimator for $\theta$ and

$$var(\hat{\theta}) = \frac{[g'(\theta)]^2}{n I_x(\theta)}$$

Then we say that $\hat{\theta}$ is a minimum variance unbiased estimator for $(\theta)$ (m.v.u.e)

$\therefore$ from the above example

$\because$ $E(\bar{x}) = \theta$ and $var(\bar{X}) = \dfrac{1}{n I_x(\theta)}$

$\therefore$ $\bar{X}$ is a m.v.u.e for $\theta$.

* Example:

Let $X_1, X_2, \dots, X_n$ be a r.s. of size $n$ from $N(\theta, \sigma^2)$ and let $\bar{X}$ be an estimator for $\theta$. Find C.R.L.B for $var(\bar{X})$.

Solu.:

$$f(x;\theta,\sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x-\theta)^2}$$

$$\ln f(x;\theta,\sigma^2) = \ln\left[\frac{1}{\sqrt{2\pi}\,\sigma}\right] - \frac{1}{2\sigma^2}(x-\theta)^2$$

$$\frac{\partial \ln f(x;\theta,\sigma^2)}{\partial \theta} = \frac{-2}{2\sigma^2}(x-\theta)(-1) = \frac{x-\theta}{\sigma^2} = \frac{x}{\sigma^2} - \frac{\theta}{\sigma^2}$$

$$\frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2} = \frac{-1}{\sigma^2}$$

$$I_x(\theta) = -E\left[\frac{-1}{\sigma^2}\right] = \frac{1}{\sigma^2}$$

Fisher information in $n$ random observations

$$= n I_x(\theta) = \frac{n}{\sigma^2}$$

From C.R. inequality we have:

$$Var(T) \geq \frac{[g'(\theta)]^2}{n I_x(\theta)}$$

$$T = \bar{X}, \ g(\theta) = \theta \Longrightarrow g'(\theta) = 1$$

$$\text{C.R.L.B for } var(\bar{x}) = \frac{1}{n I_x(\theta)} = \frac{1}{\frac{n}{\sigma^2}} = \frac{\sigma^2}{n}$$

* from the above example is $\bar{X}$ a m.v.u.e for $\theta$?

Solu.: $\because x \sim N(\theta, \sigma^2)$

$$\Rightarrow \bar{X} \sim N\left(\theta, \frac{\sigma^2}{n}\right) \Longrightarrow var(\bar{X}) = \frac{\sigma^2}{n} = \text{C.R.L.B}$$

and $E(\bar{x}) = E\left[\frac{\sum x_i}{n}\right] = \frac{\sum E(x_i)}{n} = \frac{n\theta}{n} = \theta$ (unbiased)

$\therefore \bar{X}$ is a m.v.u.e for $\theta$.

* *Example:*

Let $x_1, x_2, \cdots, x_n$ be a r.s. of size $n$ from poisson distribution with parameter $\theta$ and let $\bar{X}$ be an estimator for $\theta$. Is $\bar{X}$ a m.v.u.e for $\theta$?

Solu:

$$f(x;\theta) = \frac{e^{-\theta}\,\theta^{x}}{x!}$$

$$\ln f(x;\theta) = -\theta + x\ln\theta - \ln x!$$

$$\frac{\partial \ln f(x;\theta)}{\partial\theta} = -1 + \frac{x}{\theta}$$

$$\frac{\partial^2 \ln f(x;\theta)}{\partial\theta^2} = \frac{-x}{\theta^2}$$

$$\Rightarrow I_x(\theta) = -E\left[\frac{-x}{\theta^2}\right] = \frac{Ex}{\theta^2} = \frac{\theta}{\theta^2} = \frac{1}{\theta}$$

$\Rightarrow$ Fisher information in $n$ random observations

$$n I_x(\theta) = \frac{n}{\theta}$$

$$T = \bar{x},\ g(\theta) = \theta \Longrightarrow g'(\theta) = 1$$

$\Rightarrow$ C.R.L.B for variance of $\bar{x} = \dfrac{1}{nI_x(\theta)} = \dfrac{1}{\frac{n}{\theta}} = \dfrac{\theta}{n}$

$$Var(\bar{x}) = V\left(\frac{\sum x_i}{n}\right) = \frac{n\,var(x_i)}{n^2} = \frac{n\theta}{n^2} = \frac{\theta}{n} = \text{C.R.L.B for } var(\bar{x})$$

$$\because E(\bar{x}) = E\left[\frac{\sum x_i}{n}\right] = \frac{nEx_i}{n} = \frac{n\theta}{n} = \theta \text{ (unbiased)}$$

$\therefore \bar{X}$ is a m.v.u.e for $\theta$.