* The Maximum Likelihood Method :-

This is another method for estimating the parameter θ. The Concept of this method depends on the likelihood function which can be defined as follows:-

Def:
Let $\{X_i\}$, $i=1,2,\dots,n$ be a r.s. from distribution that has a p.d.f. $f(x;\theta)$, $\theta \in \Omega$. Then the likelihood function L is the joint probability density function of the random sample $x_1, x_2, \dots, x_n$

$$L(\theta; x_1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n; \theta)$$
$$= \prod_{i=1}^{n} f(x_i;\theta), \quad \text{if } x_1, x_2, \dots, x_n \text{ are independent random observations}$$

Method of Maximum Likelihood Estimation (m.l.e)

The method is to choose an estimate $\hat{\theta}$ (value of the parameter θ) that maximizes the likelihood function $L(\theta)$: that is

$$L(\hat{\theta}; x_1, \dots, x_n) = \max\{L(\theta; x_1, \dots, x_n), \theta \in \Omega\}$$

Then $\hat{\theta}$ is called a m.l.e. of θ

For finding out maximum likelihood estimator for the unknown parameter θ, it will be the solution of the following equation:

$$\frac{\partial L(\theta)}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial^2 L(\theta)}{\partial \theta^2} < 0$$

It was noticed that it is easier to use logarithm of the likelihood function. Furthermore, $\ln L(\theta)$ and $L(\theta)$ have their maximum at the same value of θ. Therefore, we can have an estimate for the parameter θ as a solution of the equation

$$\frac{\partial \ln L(\theta; x_1, \dots, x_n)}{\partial \theta} = 0$$

* Example:-

Let $x_1, x_2, \dots, x_n$ be a r.s. from the Bernoulli distribution with parameter p. Estimate p by using the maximum likelihood method.

Solu:

$X \sim Ber(p)$ or $X \sim bin(1, p)$

$$f(x;p) = \begin{cases} p^x (1-p)^{1-x} & x = 0, 1 \\ 0 & o.w. \end{cases}$$

$$L(p) = \prod_{i=1}^{n} f(x_i; p)$$

$$= \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^{\sum_{i=1}^{n} x_i} (1-p)^{n - \sum_{i=1}^{n} x_i}$$

$$\ln L(p) = \sum_{i=1}^{n} x_i \ln p + \left(n - \sum_{i=1}^{n} x_i\right) \ln(1-p)$$

$$\frac{\partial \ln L(p)}{\partial p} = \frac{\sum_{i=1}^{n} x_i}{p} - \frac{n-\sum_{i=1}^{n} x_i}{1-p} = 0 \quad ---- (1)$$

$$\frac{\partial^2 \ln L(p)}{\partial p^2} = \frac{-\sum_{i=1}^{n} x_i}{p^2} - \frac{n-\sum x_i}{(1-p)^2} = \frac{-\sum x_i (1-p)^2 - p^2(n-\sum x_i)}{p^2(1-p)^2}$$

$$= \frac{-\sum x_i (1-2p+p^2) - p^2 n + p^2 \sum x_i}{p^2(1-p)^2}$$

$$= \frac{-\sum x_i + 2p\sum x_i - p^2 \cancel{\sum x_i} - np^2 + p^2 \cancel{\sum x_i}}{p^2(1-p)^2} < 0$$

وبحل المعادلة رقم (1)

$$\frac{\sum_{i=1}^{n} x_i}{p} = \frac{n-\sum_{i=1}^{n} x_i}{1-p} \Longrightarrow (1-p)\sum_{i=1}^{n} x_i = p(n-\sum_{i=1}^{n} x_i)$$

$$\Longrightarrow \sum x_i - p\cancel{\sum x_i} = np - p\cancel{\sum x_i}$$

$$\Longrightarrow \hat{p} = \frac{\sum x_i}{n} = \bar{X}$$

$\therefore \bar{X}$ is a m.l.e for the parameter p.

Example:

Let $X_1, X_2, ---, X_n$ be independent identically distributed (i.i.d) r.v. with p.d.f

$$f(x;\theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & x > 0 \\ 0 & o.w. \end{cases}$$

Use the m.l.e to estimate the parameter θ

Solu:

$$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{x_i}{\theta}} = \left(\frac{1}{\theta}\right)^n e^{-\frac{\sum x_i}{\theta}}$$

$$\ln L(\theta) = \underbrace{n \ln 1}_{\text{zero}} - n \ln \theta - \frac{\Sigma x_i}{\theta}$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{-n}{\theta} + \frac{\Sigma x_i}{\theta^2} = 0 \quad , \quad \frac{\partial^2 \ln L(\theta)}{\partial \theta^2} < 0$$

$$\Rightarrow \frac{n}{\theta} = \frac{\Sigma x_i}{\theta^2} \Rightarrow \theta \Sigma x_i = n\theta^2 \Rightarrow \Sigma x_i = n\theta$$

$$\rightarrow \hat{\theta} = \frac{\Sigma x_i}{n} = \bar{x}$$

$\therefore \bar{x}$ is the m.l.e for the parameter θ.

*Example:

Find the m.l.e for the parameter of binomial distribution

solu:

$$f(x;p) = C_x^n \, p^x (1-p)^{n-x} \qquad x = 0, 1, 2, \ldots, n$$

$$L(p) = C_x^n \, p^x (1-p)^{n-x}$$

لانه توزيع binomial يمثل n من محاولات برنولي

$$\ln L(p) = \ln C_x^n + x \ln p + (n-x) \ln(1-p)$$

$$\frac{\partial \ln L(p)}{\partial p} = 0 + \frac{x}{p} - \frac{n-x}{1-p}$$

To find the m.l.e we solve

$$\frac{\partial \ln L(p)}{\partial p} = 0 \Rightarrow \frac{x}{p} - \frac{n-x}{1-p} = 0 \Rightarrow \frac{x}{p} = \frac{n-x}{1-p}$$

$$\Rightarrow x(1-p) = p(n-x) \Rightarrow x \cancel{- xp} = np \cancel{- xp}$$

$$\Rightarrow \hat{p} = \frac{x}{n}$$

$\therefore \frac{x}{n}$ is the m.l.e for p

Remark:

Consider the random sample $X_1, X_2, \dots, X_n$ whose distribution has the p.d.f. $f(x; \theta_1, \dots, \theta_k)$. Therefore, the likelihood function L contains k parameters that is:-

$$L(\theta_1, \theta_2, \dots, \theta_k ; x_1, \dots, x_n) = \prod_{i=1}^{n} f(x_i ; \theta_1, \dots, \theta_k)$$

Then the m.l.e is the solution of these equations:

$$\frac{\partial L}{\partial \theta_1} = 0 \quad , \quad \frac{\partial L}{\partial \theta_2} = 0 \quad , \dots , \quad \frac{\partial L}{\partial \theta_k} = 0$$

* **Example:**

Let $X_1, X_2, \dots, X_n$ be a r.s. from $N(\mu, \sigma^2)$. Find the m.l.e for μ (σ = known)

Solu:-

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$$L(\mu) = \prod_{i=1}^{n} f(x_i; \mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2}$$

$$= \frac{1}{(\sqrt{2\pi})^n \sigma^n} e^{-\frac{1}{2\sigma^2}\Sigma(x_i-\mu)^2}$$

$$\ln L(\mu) = \ln\left(\frac{1}{(\sqrt{2\pi})^n \sigma^n}\right) - \frac{1}{2\sigma^2}\Sigma(x_i-\mu)^2$$

$$\frac{\partial \ln L(\mu)}{\partial \mu} = 0 - \frac{2\,\Sigma(x_i-\mu)(-1)}{2\sigma^2} \Longrightarrow \frac{\sum_{i=1}^{n}(x_i-\mu)}{\sigma^2} = 0$$

$$\Longrightarrow \sum_{i=1}^{n} x_i - n\mu = 0 \Longrightarrow \hat{\mu} = \frac{\Sigma x_i}{n} = \overline{X} \quad (\text{m.l.e for } \mu)$$

*Example:

Let $x_1, x_2, \ldots, x_n$ be a r.s. from the normal population with mean μ and variance σ^2. use the m.l.e method to estimate μ and σ^2.

solu:

The likelihood function of the random sample $x_1, x_2, \ldots, x_n$ is :-

$$L(\mu, \sigma^2; x_1, \ldots, x_n) = \prod_{i=1}^{n} f(x_i; \mu, \sigma^2)$$

$$= \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n e^{-\frac{\Sigma(x_i-\mu)^2}{2\sigma^2}}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{\Sigma(x_i-\mu)^2}{2\sigma^2}}$$

$$\ln L(\mu, \sigma^2) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\Sigma(x_i-\mu)^2$$

$$\frac{\partial \ln L(\mu,\sigma^2)}{\partial \mu} = -\frac{2\,\Sigma(x_i-\mu)(-1)}{2\sigma^2} \Rightarrow \frac{\sum_{i=1}^{n}(x_i-\mu)}{\sigma^2} = 0 \quad \text{--- (1)}$$

$$\frac{\partial \ln L(\mu,\sigma^2)}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2 = 0 \quad \text{--- (2)}$$

the solution of these two equations is :

بحل المعادلة الاولى وتعويضها في المعادلة الثانية نحصل على :-

$$\frac{1}{\sigma^2}\Sigma(x_i-\mu) = 0 \Rightarrow \Sigma x_i - n\mu = 0 \Rightarrow \hat{\mu} = \frac{\Sigma x_i}{n} = \bar{X}$$

$$\frac{n}{2\sigma^2} = \frac{\Sigma(x_i-\mu)^2}{2\sigma^4} \Rightarrow n = \frac{\sum(x_i-\bar{x})^2}{\sigma^2} \Rightarrow \hat{\sigma}^2 = \frac{\Sigma(x_i-\bar{x})^2}{n}$$

Therefore, $\hat{\mu}$ and $\hat{\sigma}^2$ are the m.l.e for the parameters μ and σ^2 respectively.

* **Example:**

Find the m.l.e for σ^2 (θ = known) if $x_1, \dots, x_n$ is a r.s. of size n from $N(\theta, \sigma^2)$

Solu:

$$f(x; \theta, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x_i-\theta)^2}$$

$$L(\sigma^2) = \prod_{i=1}^{n} f(x_i; \theta, \sigma^2)$$

$$= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta)^2}$$

$$\ln L(\sigma^2) = \frac{-n}{2}\ln(2\pi) - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum(x_i-\theta)^2$$

$$\frac{\partial \ln L(\sigma^2)}{\partial \sigma^2} = 0 - \frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum(x_i-\theta)^2$$

To find the m.l.e for σ^2 we solve the equation

$$\frac{\partial \ln L(\sigma^2)}{\partial \sigma^2} = 0 \Rightarrow \frac{-n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum(x_i-\theta)^2 = 0$$

$$\Rightarrow \frac{n}{2\sigma^2} = \frac{1}{2\sigma^4}\sum(x_i-\theta)^2$$ نضرب الطرفين في $2\sigma^2$

$$\Rightarrow n = \frac{\sum(x_i-\theta)^2}{\sigma^2}$$

$$\Rightarrow \hat{\sigma}^2 = \frac{\sum(x_i-\theta)^2}{n}$$

$\therefore \frac{\sum(x_i-\theta)^2}{n}$ is the m.l.e for σ^2