Ministry of Higher Education and Scientific Research
University of Mosul
College of Computer Science and Mathematics
Mathematics Department
Fourth Stage

1

COMPLEX ANALYSIS

1st Course

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Dr. MOHAMMED SABAH AL TAEE

Def: let $z = x + yi$, then we can write the number $z = (x, y)$ as order pair .

If $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ then $z_1 = z_2 \leftrightarrow x_1 = x_2 \; and \; y_1 = y_2$

The operation are the complex variables

$z_1 \mp z_2 = (x_1, y_1) \mp (x_2, y_2) = (x_1 \mp x_2 , y_1 \mp y_2)$

$z_1 * z_2 = (x_1, y_1) * (x_2, y_2) = (x_1x_2 - y_1y_2 , x_1y_2 + x_2y_1)$

$$\frac{z_1}{z_2} = \frac{x_1 + y_1 i}{x_2 + y_2 i} * \frac{x_2 - y_2 i}{x_2 - y_2 i} = \frac{(x_1x_2 + y_1y_2)}{x_2^2 + y_2^2} + \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}$$

Regular form of the Complex Number

is a complex number of the representation $z = x + yi$ such that x is the real part Re(z) and written first and y is the imaginary part Im(z) and written last as $3 - 2i$

Purely Complex Number

The complex number is called **purely real** if the imaginary part is zero .

The complex number is called purely **real Imaginary** if the real part is zero .

$4 - 7i$ is a complex number , 7 is purely real , $5i$ is purely imaginary

Conjugate Complex Number

The conjugate complex number of the complex number $z = x + yi$ is $\bar{z} = x - yi$

$z = 5 + 2i \Rightarrow \bar{z} = 5 - 2i$

$z = 6 - 7i \Rightarrow \bar{z} = 6 + 7i$

$z = 2i \Rightarrow \bar{z} = -2i$

$z = 2i \Rightarrow \bar{z} = 2i$

Note if z is a complex number , $\bar{z}$ is its conjugate , then :

- $z + \bar{z} = 2\,Re(z)$
- $z - \bar{z} = 2i\,Im(z)$
- $z\bar{z} = Re^2(z) + Im^2(z)$

The absolute value properties

1. $|z| = \sqrt{z\,\bar{z}}$

$$|z| = \sqrt{x^2 + y^2} = \sqrt{Re^2(z) + Im^2(z)} = \sqrt{z\bar{z}}$$

2. $|z_1 z_2| = |z_1|\,|z_2|$

$$|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2})$$
$$= z_1\bar{z}_1 \,.\, z_2\bar{z}_2$$
$$= |z_1|^2\,|z_2|^2$$
$$= (|z_1|\,|z_2|)^2$$
$$\therefore\ |z_1 z_2| = |z_1|\,|z_2|$$

3. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$

$$\left|\frac{z_1}{z_2}\right|^2 = \left(\frac{z_1}{z_2}\right)\overline{\left(\frac{z_1}{z_2}\right)} \quad , z_2 \neq 0$$
$$= \left(\frac{z_1}{z_2}\right)\frac{\bar{z}_1}{\bar{z}_2} = \frac{z_1\bar{z}_1}{z_2\bar{z}_2} = \frac{|z_1|^2}{|z_2|^2} = \left(\frac{|z_1|}{|z_2|}\right)^2$$
$$\therefore\ = \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

4. $|z_1 + z_2| \leq |z_1| + |z_2|$

$$|z_1 + z_2|^2 = (z_1 + z_2)\,\overline{(z_1 + z_2)}$$
$$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$$

$$= z_1\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_2} + z_2\overline{z_1}$$

Let $u = z_1\overline{z_2} \rightarrow \bar{u} = \overline{z_1}z_2$

$$|z_1 + z_2|^2 = z_1\overline{z_1} + u + \bar{u} + z_2\overline{z_2}$$

$$\because u + \bar{u} = 2Re(u) \leq 2|u| = 2|z_1\overline{z_2}|$$

$$\therefore |z_1 + z_2|^2 \leq |z_1|^2 + 2|z_1\overline{z_2}| + |z_2|^2$$

$$|z_1 + z_2|^2 \leq |z_1|^2 + 2|z_1||z_2| + |z_2|^2$$

$$|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$$

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

5. $\mathbf{|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|}$

H.W

6. $|\mathbf{z}| = |\bar{\mathbf{z}}|$

$|z| = \sqrt{z\,\bar{z}}$

$|z|^2 = z\,\bar{z} = \bar{z}\,z \ = |\bar{z}|^2$

$\therefore\ |z| = |\bar{z}|$

7. $|\mathbf{z_1} - \mathbf{z_2}| \ \geq |\mathbf{z_1}| - |\mathbf{z_2}|$

H.W

Example :

prove $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,(|z_1|^2 + |z_2|^2)$

Solution

$|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)}$

$= (z_1 + z_2)(\bar{z_1} + \bar{z_2})$

$= z_1\bar{z_1} + z_1\bar{z_2} + z_2\bar{z_1} + z_2\bar{z_2}$ ... (1)

$|z_1 - z_2|^2 = (z_1 - z_2)\overline{(z_1 - z_2)}$

$= (z_1 - z_2)(\bar{z_1} - \bar{z_2})$

$= z_1\bar{z_1} - z_1\bar{z_2} - z_2\bar{z_1} + z_2\bar{z_2}$ ... (2)

By adding (1) and (2) we get

$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2z_1\bar{z_1} + 2z_2\bar{z_2}$

$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(\,|z_1|^2 + |z_2|^2)$

Example :

The number z be purely real or purely imaginary if $(\bar{z}^2) = z^2$

Solution

Let $z = x + yi \;\rightarrow\; \bar{z} = x - yi$

$(x + yi)^2 = (x - yi)^2$

$x^2 + 2xyi - y^2 = x^2 - 2xyi - y^2$

$4xyi = 0 \;\rightarrow xy = 0$

$either \quad x = 0 \;\rightarrow\; imaginary\ purely$

$or \quad y = 0 \;\rightarrow\; real\ purely$

Geometrical Representation of Complex Number

Example :

sketch (represent) the graph of the following curves

I. $|z-i| = |z+i|$

Solution

$$|x+yi-i| = |x+yi+i|$$

$$|x+(y-1)i| = |x+(y+1)i|$$

$$x^2+(y-1)^2 = x^2+(y+1)^2$$

$$y^2-2y+1 = y^2+2y+1$$

$$4y = 0 \rightarrow y = 0$$

The curve is represent the x-axis

II. $|z-4i| + |z+4i| = 10$

Solution

$$|x+yi-4i| + |x+yi+4i| = 10$$

$$|x+(y-4)i| + |x+(y+4)i| = 10$$

$$\sqrt{x^2+(y-4)^2} + \sqrt{x^2+(y+4)^2} = 10$$

$$\sqrt{x^2+(y-4)^2} = 10 - \sqrt{x^2+(y+4)^2}$$, Squared the both sides

$$x^2+y^2-8y+16 = 100 - 20\sqrt{x^2+(y+4)^2} + x^2+y^2+8y+16$$

$$-16y-100 = -20\sqrt{x^2+y^2+8y+16}$$

$$4y+25 = 5\sqrt{x^2+y^2+8y+16}$$

$$16y^2+200y+625 = 25(x^2+y^2+8y+16)$$

$$16y^2 + 200y + 625 = 25x^2 + 25y^2 + 200y + 400$$

$$9x^2 + 25y^2 = 225 \qquad \div (25)$$

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

is an equation of ellipse its focus on the x-axis

III. $z\bar{z} + \bar{b}z + b\bar{z} = 0 \quad , \quad b = (-h, -k)$

$Let \quad z = x + yi \ , \bar{z} = x - yi$

$(x + yi)(x - yi) + \bar{b}(x + yi) + b(x - yi) = b\bar{b}$

$x^2 + y^2 + \bar{b}x + \bar{b}yi + bx - byi = 0$

$x^2 + (\bar{b} + b)x + y^2 + (\bar{b} - b)yi = 0$ such that $b = (-h, -k), \bar{b} = (-h, k)$

$\because \ \bar{b} + b = -2h \quad , \quad \bar{b} - b = 2ki$

$x^2 - 2hx + y^2 + 2ki\, yi = b\bar{b} \quad , \quad i^2 = -1$

$x^2 - 2hx + y^2 - 2k\, y = b\bar{b}$

$x^2 - 2hx + h^2 + y^2 - 2k\, y + k^2 = h^2 + k^2$

$(x - h)^2 + (y - k)^2 = r^2$ such that $h^2 + k^2 = r^2$

is an equation of circle its center is (h, k) and radius is $r = \sqrt{h^2 + k^2}$

IV. $\left|\frac{z-(1+i)}{z-(2+3i)}\right| = 1$

H.W

The modulus & Argument

We can define the **modulus** of any number $z = x + yi$ as

$$|z| = \sqrt{x^2 + y^2} \quad \geq 0$$

The modulus always positive

The argument can be found by

$$\tan\theta = \frac{y}{x} \quad or \quad \theta = \tan^{-1}\frac{y}{x}$$

Such that θ is the argument and it's value being $0 \leq \theta \leq 2\pi$

To evaluate the argument of any no. z

We must know the types of angles

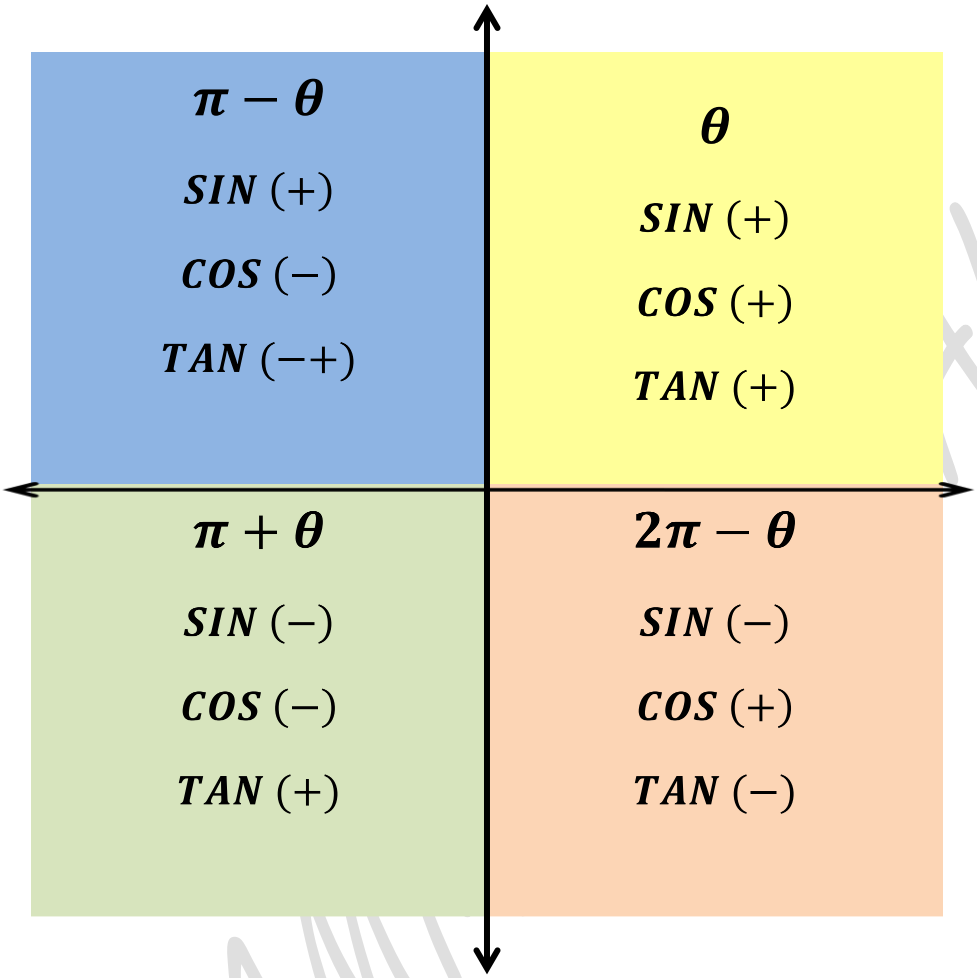
(special – complementary – and periodic)

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	$-\infty$	0

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$
$$\sin(\pi - \theta) = \sin\theta$$
$$\sin(\pi + \theta) = -\sin\theta$$
$$\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos\theta$$
$$\sin(2\pi - \theta) = -\sin\theta$$

REVIEW

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$
$$\cos(\pi - \theta) = \cos\theta$$
$$\cos(\pi + \theta) = -\cos\theta$$
$$\cos\left(\frac{3\pi}{2} - \theta\right) = -\cos\theta$$
$$\cos(2\pi - \theta) = \cos\theta$$



Example :

Find the modulus & argument to the following complex numbers

Solution

I. $\boldsymbol{z = 1 + i}$

$x = 1\ ,\ y = 1\quad ,(1,1) \in 1st\ \ Q.$
$r = \sqrt{1+1} = \sqrt{2}$

$$\theta = \tan^{-1}\frac{1}{1} \ \rightarrow \theta = \tan^{-1}(1)$$

$$\theta = \frac{\pi}{4} \ \ \rightarrow \ \ arg\ z = \frac{\pi}{4}$$

II. $z = \sqrt{3} - 3i$

$x = \sqrt{3}\ ,\ y = -3 \quad ,(\sqrt{3}, -3) \in 4th\ \ Q.$

$r = \sqrt{3+9} = \sqrt{12} = 2\sqrt{3}$

$\theta = \tan^{-1}\frac{-3}{\sqrt{3}} \ \rightarrow \theta = \tan^{-1}(-\sqrt{3})$

$\theta = \frac{\pi}{3} \quad \rightarrow \quad arg\ z = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

III. $z = -3\sqrt{3} - 3i$

H.W

IV. $z = -2 + 2i$

H.W

Argument Properties

- $Arg(z_1\,z_2) = Arg(z_1) + Arg(z_2)$
- $Arg(z_1\,z_2\,...\,z_n) = Arg(z_1) + Arg(z_2) + \cdots + Arg(z_n)$
- $Arg\left(\frac{z_1}{z_2}\right) = Arg(z_1) - Arg(z_2)$
- $Arg(\bar{z}) = -Arg(z)$
- $Arg(z^n) = n\,Arg(z)$

Example :

if $z_1 = 3 + 3i$ and $z_2 = 1 - \sqrt{3}i$, find the value of :

$Arg\,z_1$, $Arg\,z_2$, $Arg(z_1\,z_2)$, $Arg\left(\frac{z_1}{z_2}\right)$, $Arg(\bar{z}_1)$, $Arg(z_1^5\,)$

Solution

$$Arg\,z_1 = \tan^{-1}\frac{3}{3} = = \tan^{-1}1 = \frac{\pi}{4}$$

$$Arg\,z_2 = \tan^{-1}\frac{\sqrt{3}}{1} = = \tan^{-1}\sqrt{3} = \frac{\pi}{3} \Rightarrow \quad Arg\,z_2 = \frac{5\pi}{3}$$

$$Arg(z_1\,z_2) = Argz_1 + Argz_2 = \frac{\pi}{4} + \frac{5\pi}{3} = \frac{23\pi}{12}$$

$$Arg\left(\frac{z_1}{z_2}\right) = Argz_1 - Argz_2 = \frac{\pi}{4} - \frac{5\pi}{3} = \frac{7\pi}{12}$$

$$Arg(\bar{z}_1) = -Arg(z_1) = -\frac{\pi}{4}\,or\,\frac{7\pi}{4}$$

$$Arg({z_1}^5\,) = 5\,Arg(z_1\,) = 5\left(\frac{\pi}{4}\right) = \frac{5\pi}{4}$$

Angles Types (Review)

to find the argument (angle) for any complex number then :

There are three types of Angles :

A. Special Angles

it's the direct angles ($0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$)

and calculated easily .

B. Complementary Angles

it is the angle between $0 \leq \vartheta \leq 2\pi$ and calculated by the sign of the trigonometric function and the location (value) of its angle

$$\sin\frac{5\pi}{6} = \sin\frac{\pi}{6} = \frac{1}{2}$$

$$\cos\frac{5\pi}{4} = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$\sin\frac{5\pi}{3} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

C. Periodic Angles

is angle that is measure larger than $360°$ or 2π there are three types :

- **Odd Periodic Angle**

  same as $\pm\pi, \pm 3\pi, \pm 5\pi$, ... and the result from this angle is always equal to π

  $\cos(23\pi) = \cos(\pi) = -1$

- **Even Periodic Angle**

  same as $0, \pm 2\pi, \pm 4\pi, \pm 6\pi$, ... and the result from this angle is always equal to 0

  $\sin(8\pi) = \sin 0 = 0$