❖ Fractional Periodic Angle

is angle of the representation $\frac{a}{b}\pi$ such that $\frac{a}{b} > 2$ and the result from this angle can be calculated from the operation : $(a) \div (2b)$, this operation having result and remainder , Geometrically the result represent the complete period of angle , and the remainder represent the un-complete period of this angle , the remainder is the importance value , we will take it and replace it by the value of a in the original angle.

$$\sin\frac{49\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\cos\left(\frac{23\pi}{3}\right) = \cos\left(\frac{5\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$\cos\left(\frac{29\pi}{6}\right) = \cos\left(\frac{5\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$

The Power of Complex Number (De moivre's theorem)

The complex number power formula is used to compute the value of a complex number which is raised to the power of "n". To recall, a complex number is the form of x + iy, where x and y are the real numbers and is an imaginary number.

A. Case 1 : when n is an integer number

let $z = x + yi$ to calculate z^n, when n is an integer we use the euler formula : $e^{i\theta} = \cos\theta + i\sin\theta$

$z = re^{i\theta} \Rightarrow z^n = \left(re^{i\theta}\right)^n$, r is modulus , θ is argument , n power

$$\boxed{z^n = r^n\, e^{in\theta}}$$

Example :

find the value of $(3+\sqrt{3}i)^{100}$

Solution

$x = 3 \quad , y = \sqrt{3} \quad , n = 100 \quad , (3, \sqrt{3}) \in 1st\ .Q$

$r = \sqrt{9+3} = \sqrt{12} = 2\sqrt{3}$

$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{\sqrt{3}}{3} = \tan^{-1}\frac{1}{\sqrt{3}} \rightarrow \theta = \frac{\pi}{6}$

$z^n = r^n e^{in\theta}$

$z^{100} = r^{100}\,.\,e^{i\,100\,\theta}$

$= (2\sqrt{3})^{100}\,.\,e^{i\,100\frac{\pi}{6}}$

$= 2^{100}.\,3^{50}\,.\,e^{i\,50\frac{\pi}{3}}$

$= 2^{100}.\,3^{50}\,.\,e^{i\,\frac{2\pi}{3}}$

$= 2^{100}.\,3^{50}\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]$

$= 2^{100}.\,3^{50}\left[-\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$

$= 2^{100}.\,3^{50}\left[-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right]$

$= 2^{99}.\,3^{50}(-1+\sqrt{3}\,i)$

Example :

find the value of $(2-2i)^{21}$

Solution:

$x=2 \quad ,y=-2 \quad ,n=100 \quad ,(x,y)\in 4th\,.Q$

$r=\sqrt{4+4}=\sqrt{8}=2\sqrt{2}$

$\theta=\tan^{-1}\frac{y}{x}=\tan^{-1}\left(\frac{-2}{2}\right)=\tan^{-1}(-1)\rightarrow\theta=\frac{\pi}{4}$

$\theta=\frac{\pi}{4}\quad\rightarrow\quad\theta=2\pi-\frac{\pi}{4}=\frac{7\pi}{4}$

$z^{n}=r^{n}e^{in\theta}$

$z^{21}=(2\sqrt{2})^{21}\,.\,e^{i\,21.\frac{7\pi}{4}}$

$=(2\sqrt{2})^{21}\left[e^{i\,\frac{147\pi}{4}}\right]$

$=(2\sqrt{2})^{21}\left[e^{i\,\frac{3\pi}{4}}\right]$

$=2^{21}.2^{10}.\sqrt{2}\left[\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right]$

$=2^{31}.\sqrt{2}\left[-\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right]$

$=2^{31}.\sqrt{2}\left[-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right]$

$=2^{31}[-1+i]$

Example :

find the value of $\left(-\sqrt{2}+\sqrt{6}i\right)^{23}$

H.W

B. Case 2 : when n is a radical number (root)

the n-th root of the complex number z is given by

$$w_k = r^{\frac{1}{n}}\left[\cos\frac{\theta+2\pi k}{n} + i\sin\frac{\theta+2\pi k}{n}\right]$$

such that r is the modulus , θ is argument , n is the order Root of z

$k = 0,1,2,\dots,n-1$

Notes :

- The number of solution are equal to the order of z .
- if the angle as $\frac{m\pi}{2,3,4,6}$, then we simplify the function and find the numerical value .if not , the angle stay .
- if the numbers of roots larger than two roots , then we can estimate the roots (from 3rd to the last) by distance numerator ز
- if the number is purely , then we calculate the modulus and argument directly from $z = r[\cos\theta + i\sin\theta]$

$1 = \cos 0 + i\sin 0$

$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$-1 = \cos\pi + i\sin\pi$

$-i = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$

$4 = 4(1) = 4(\cos 0 + i\sin 0) \rightarrow r = 4\ ,\theta = 0$

$2i = 2(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) \rightarrow r = 2\ ,\theta = \frac{\pi}{2}$

$-3 = 3(-1) = 3(\cos\pi + i\sin\pi) \rightarrow r = 3\ ,\theta = \pi$

$-5i = 5(-i) = 5\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) \rightarrow r = 5\ ,\theta = \frac{3\pi}{2}$

Example :

Find the square root of $z = 3\sqrt{3} + 3i$

Solution

$x = 3\sqrt{3}$, $y = 3$, $(3\sqrt{3}, 3) \in lst.Q$

$n = 2$, $k = 0,1$

$r = \sqrt{27 + 9} = 6$

$$\theta = \tan^{-1}\frac{3}{3\sqrt{3}} = \tan^{-1}\frac{1}{\sqrt{3}} \rightarrow \theta = \frac{\pi}{6}$$

$$w_k = r^{\frac{1}{n}}\left[\cos\left(\frac{\frac{\pi}{6} + 2\pi k}{2}\right) + i\sin\left(\frac{\frac{\pi}{6} + 2\pi k}{2}\right)\right]$$

$k = 0$

$$w_0 = \sqrt{6}\left[\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right]$$

$k = 1$

$$w_1 = \sqrt{6}\left[\cos\frac{\frac{\pi}{6} + 2\pi(1)}{2} + i\sin\frac{\frac{\pi}{6} + 2\pi(1)}{2}\right]$$

$$w_1 = \sqrt{6}\left[\cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}\right]$$

Example :

Find the cubic root of $z = -27i$

Solution

$$-27i = 27(-i) = 27\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$

$$r = 27 \ ,\theta = \frac{3\pi}{2} \ ,n = 3 \ ,k = 0,1,2$$

$$w_k = (27)^{\frac{1}{3}}\left[\cos\frac{\frac{3\pi}{2} + 2\pi k}{3} + i\sin\frac{\frac{3\pi}{2} + 2\pi k}{3}\right]$$

$k = 0$

$$w_0 = 3\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right] = 3i$$

$k = 1$

$$w_1 = 3\left[\cos\frac{\frac{3\pi}{2} + 2\pi}{3} + i\sin\frac{\frac{3\pi}{2} + 2\pi}{3}\right]$$

$$w_1 = 3\left[\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right]$$

$$w_1 = 3\left[-\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right] = 3\left[-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right]$$

$k = 2$

$$w_2 = 3\left[\cos\frac{\frac{3\pi}{2} + 4\pi}{3} + i\sin\frac{\frac{3\pi}{2} + 4\pi}{3}\right] = 3\left[\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right]$$

$$w_2 = 3\left[\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right] = 3\left[\frac{\sqrt{3}}{2} - \frac{1}{2}i\right]$$

Example :

Solve the equation $4z^3 + 28 = 0$, $z \in \mathbb{C}$

Solution

$4z^3 = -28 \rightarrow z^3 = -7$

$z = \sqrt[3]{-7}$

$-7 = 7(-1) = 7(\cos\pi + i\sin\pi)$

$r = 7 \ ,\theta = \pi \ ,n = 3 \ ,k = 0,1,2$

$$w_k = (7)^{\frac{1}{3}}\left[\cos\frac{\pi + 2\pi k}{3} + i\,sin\frac{\pi + 2\pi k}{3}\right]$$

$k = 0$

$$w_0 = \sqrt[3]{7}\left[\cos\frac{\pi}{3} + isin\frac{\pi}{3}\right] = \sqrt[3]{7}\left[\frac{1}{2} + \frac{\sqrt{3}}{2}i\right]$$

$k = 1$

$$w_1 = \sqrt[3]{7}\left[\cos\frac{\pi + 2\pi}{3} + isin\frac{\pi + 2\pi}{3}\right]$$

$$w_1 = \sqrt[3]{7}[\cos\pi + isin\,\pi] = \sqrt[3]{7}[-1 + 0i] = -\sqrt[3]{7}$$

$k = 2$

$$w_2 = \sqrt[3]{7}\left[\cos\frac{\pi + 4\pi}{3} + isin\frac{\pi + 4\pi}{3}\right]$$

$$w_2 = \sqrt[3]{7}\left[\cos\frac{5\pi}{3} + isin\frac{5\pi}{3}\right]$$

$$w_2 = \sqrt[3]{7}\left[\cos\frac{\pi}{3} - isin\frac{\pi}{3}\right] = \sqrt[3]{7}\left[\frac{1}{2} - \frac{\sqrt{3}}{2}i\right]$$

Example :

Solve the equation $2z^4 + 30\,i^3 = 2i \quad , z \in \mathbb{C}$

Solution

$$2x^4 - 30i = 2i \;\rightarrow\; 2x^4 = 32i \;\rightarrow x^4 = 16i$$

$$x = \sqrt[4]{16i}$$

$$16i = 16\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$$

$$r = 16 \;\; ,\theta = \frac{\pi}{2} \;\; ,n = 4 \;\; ,k = 0,1,2,3$$

$$w_k = (16)^{\frac{1}{4}}\left[\cos\frac{\frac{\pi}{2} + 2\pi k}{4} + i\sin\frac{\frac{\pi}{2} + 2\pi k}{4}\right]$$

$k = 0$

$$w_0 = 2\left[\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right]$$

$k = 1$

$$w_1 = 2\left[\cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}\right]$$

$k = 2$

$$w_2 = 2\left[\cos\frac{\frac{\pi}{2} + 4\pi}{4} + i\sin\frac{\frac{\pi}{2} + 4\pi}{4}\right] = 2\left[\cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}\right]$$

$k = 3$

$$w_3 = 2\left[\cos\frac{\frac{\pi}{2} + 6\pi}{4} + i\sin\frac{\frac{\pi}{2} + 6\pi}{4}\right]$$

$w_3 = 2\left[\cos\frac{13\pi}{8} + i\sin\frac{13\pi}{8}\right]$

Example :

Solve the equation $z^5 + 32 = 0$, $z \in \mathbb{C}$

H.W

C. Case 3 : when n is a fractional (m/n)

$$\left(z^{\frac{m}{n}}\right) = \left\{ \underbrace{(z^m)}_{De\ moiver\ integer} \right\}^{1/n}$$

Notes :

- ✓ first we solve the number can raised to intger power , and from result we find the 2nd part of radical De-Moiver but by new values of modulus and argument .
- ✓ $z^{\frac{m}{n}} = h$, that is can solved to find z by : $z = h^{\frac{n}{m}}$

Example :

Solve the equation $z^{3/2} + (1 + i) = 0 \quad , z \in \mathbb{C}$

Solution

$$z^{3/2} + (1 + i) = 0 \quad , z \in \mathbb{C}$$

$$z^{3/2} = -1 - i \quad \Rightarrow z = (-1 - i)^{2/3}$$

1st Step : $(-1 - i)^2$

$$x = -1 \ , y = -1 \ , \ r = \sqrt{2} \ , \ \theta = \frac{3\pi}{4} \ , \ n = 2$$

$$z^2 = r^n e^{in\theta}$$

$$(-1 - i)^2 = \left(\sqrt{2}\right)^2 e^{i2\left(\frac{3\pi}{4}\right)} = 2\left[\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right] \to r = 2 \ , \quad \theta = \frac{3\pi}{2}$$

2nd Step :

$$r = 2 \ , \quad \theta = \frac{3\pi}{2} \ , \quad n = 3 \ , k = 0, 1, 2$$

$$[(-1 - i)^2]^{1/3} = r^{\frac{1}{n}}\left[\cos\frac{\frac{3\pi}{2} + 2\pi k}{3} + i\sin\frac{\frac{3\pi}{2} + 2\pi k}{3}\right]$$

$$(-1-i)^{\frac{2}{3}} = (2)^{1/3}[cos\ \frac{\frac{3\pi}{2}+2\pi k}{3} + i\ sin\ \frac{\frac{3\pi}{2}+2\pi k}{3}$$

$k = 0$

$$w_0 = \sqrt[3]{2}\left[\cos\frac{\pi}{2} + isin\frac{\pi}{2}\right]$$

$k = 1$

$$w_1 = \sqrt[3]{2}\left[\cos\frac{7\pi}{6} + isin\frac{5\pi}{6}\right] = \sqrt[3]{2}\left[-\cos\frac{\pi}{6} + isin\frac{\pi}{6}\right] = \sqrt[3]{2}(-\frac{\sqrt{3}}{2}+\frac{1}{2}i)$$

$k = 2$

$$w_2 = \sqrt[3]{2}\left[\cos\frac{11\pi}{6} + isin\frac{11\pi}{6}\right]$$

$$w_2 = \sqrt[3]{2}\left[\cos\frac{\pi}{6} - isin\frac{\pi}{6}\right] = \sqrt[3]{2}(\frac{\sqrt{3}}{2}-\frac{1}{2}i)$$

Example :

Solve the equation $z^{3/5} + (2-2i) = 0$, $z \in \mathbb{C}$

H.W