2023–2024

complex analysis

2nd Course

Chapter 1

Dr.Mohammed Sabah Altaee

Chapter One

Sequences and Series

The sequence

a sequence of a complex numbers is a function from the set of positive numbers N into complex numbers $\mathbb{C}$ denoted by $\{z_n\} = \{z_1, z_2, \dots \dots \dots\}$

Notes

(1) $\{z_n\}$ is called constant sequence if $z_{k+1} = z_k \ \forall \ k = 1,2, \dots \dots$

(2) $\{z_n\}$ is called increasing sequence if $z_{k+1} \geq z_k \ \forall \ k = 1,2, \dots \dots$

(3) $\{z_n\}$ is called decrasing sequence if $z_{k+1} \leq z_k \ \forall \ k = 1,2, \dots \dots$

(4) $\{z_n\}$ is bounded if $|z_n| \leq M$ Where M is $+ive$ no. , $n \in N$

(5) $\{z_n\}$ is convergent to complex no. z and written by

$\lim_{n\to\infty} z_n = z$ or $\lim_{n\to\infty} |z_n - z| = 0$

if $\{z_n\}$ not convergent then its divergent we can write it as for give $\epsilon > 0$, $\forall + ive$ integer N (may be depend on ϵ) such that

$$|z_n - z| < \epsilon \qquad \forall \ n \geq N$$

Example

is $z_n = \frac{i}{n}$ convergent or divergent ?

Solution :

$\lim_{n\to\infty} \frac{i}{n} = 0$

$\therefore\ \mathbf{z_n = \frac{i}{n}}$ **is convergent**

Example

is $\{(2i)^n\}$ convergent or divergent ?

Solution :

$\{(2i)^n\} = (2^n)(i^n)$

Since $\{i^n\} = i, -1, -i, 1$

and $\{2^n\} \to \infty\ when\ n \to \infty$

$\therefore\ z_n = (2i)^n$ is convergent

Example

let $z_n = \frac{n}{n+i}$ then show that $\{z_n\}$ is convergent to 1

Solution :

we need to show that

$\lim_{n\to\infty} |z_n - z| = 0$ [Definition of sequence convergent]

$\lim_{n\to\infty} \left|\frac{n}{n+i} - 1\right| = \lim_{n\to\infty} \left|\frac{n-n-i}{n+i}\right|$

$= \lim_{n\to\infty} \left|\frac{-i}{n+i}\right| = \lim_{n\to\infty} \frac{|-i|}{|n+i|}$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} = 0$$

$\therefore \quad \{z_n\}$ is convergent to 1

Example

let $z_n = \frac{i^n}{n}$ then show that $\{z_n\}$ is convergent to 0

Solution :

Since $\forall\ \varepsilon > 0 \quad , \exists + inv\ on. N$ such that

$\lim_{n\to\infty} |z_n - z| = 0$ [Definition of sequence convergent]

$$\lim_{n\to\infty} \left|\frac{i^n}{n} - 0\right| < \varepsilon \quad \text{when} \quad n > N$$

$$\lim_{n\to\infty} \left|\frac{i^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0$$

$\therefore \quad \{z_n\} = \frac{i^n}{n}$ is convergent to 0 $as \quad n \to \infty$

Theorem

Suppose $\{z_n\}$ is a sequence of a complex no. let

$z_n = x_n + iy_n \quad , \quad n = 1, 2, \ldots\ldots \in N$, and $z = x + iy$ then

$\lim_{n\to\infty} z_n = z$ iff $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$

Proof

$i \to ii$

Suppose $\lim_{n\to\infty} z_n = z$

$\forall\, \varepsilon > 0\ \exists\ +$ive number N Such that

$|z_n - z| < \varepsilon$ whenever $n > N$

$$\left.\begin{array}{l} \Longrightarrow |x_n + iy_n - (x + iy)| < \epsilon \\ \Longrightarrow |(x_n - x) + i(y_n - y)| < \epsilon \end{array}\right\} \ \forall\, n > N$$

But $|x_n - x| \le |(x_n - x) + i(y_n - y)| < \epsilon \ ; \quad n > N$

And also $|y_n - y| \le |(x_n - x) + i(y_n - y)| < \epsilon \ ; \quad n > N$

$\Longrightarrow |y_n - y| < \epsilon \ \ and \quad |x_n - x| < \epsilon \quad \forall\, n > N$

$\Longrightarrow \lim_{n\to\infty} x_n = x \ ; \quad \lim_{n\to\infty} y_n = y$

$ii \to i$

$let \lim_{n\to\infty} x_n = x \ ; \quad lim_{n\to\infty} y_n = y$

$\forall\, \epsilon > 0 \ \exists + ive\ \ number\ \ N_1, N_2$

Such that

$|x_n - x| < \frac{\epsilon}{2} \qquad \forall\, n > N_1$

$|y_n - y| < \frac{\epsilon}{2} \qquad \forall\, n > N_2$

Let $N = \max(N_1, N_2)$

$|z_n - z| = |x_n + iy_n - (x + iy)|$

$|z_n - z| = |(x_n - x) + i(y_n - y)|$

$$|z_n - z| \leq |x_n - x| + |y_n - y| \qquad \forall\, n > N$$

$$|z_n - z| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \forall\, n > N$$

$\therefore z_n$ is convergent to z

The series

Let z_n be a sequence of a complex no. that the sum

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots + z_n + \cdots$$

Is called a series of a complex no. where z_n is the n-th term of sequence

The partial sum (S_n)

let $S_1 = z_1$

$S_2 = z_1 + z_2$

$S_3 = z_1 + z_2 + z_3$

$\vdots$

$S_n = z_1 + z_2 + z_3 + \cdots + z_n$

$= \sum_{k=1}^{n} z_k$

S_n is called the partial sum of series but S is called the sum of all series

$$S = z_1 + z_2 + z_3 + \cdots + z_n + z_{n+1} + \cdots = \sum_{k=1}^{\infty} z_k$$

Convergent Series

The series $\sum_{n=1}^{\infty} z_n$ is convergent to S as $n \to \infty$ written as $\sum_{n=1}^{\infty} z_n = S$, if the sequence $\{S_n\}$ is convergent to S

If its not convergent it is called divergent

***Note*:-**

1. S is called the sum of the series ,which is unique since

$$\lim_{n\to\infty} S_n = S \text{ is unique .}$$

2. $S = \sum_{k=1}^{\infty} z_k = \sum_{k=1}^{n} z_k + \sum_{k=n+1}^{\infty} z_k$

$$S = S_n + R_n \text{ such that } R_n = \sum_{k=n+1}^{\infty} z_k$$

$$R_n = S - S_n$$

$$|S - S_n| = R_n$$

Then the necessary and sufficient condition to the series $\sum_{n=1}^{\infty} z_n$ to be convergent when $R_n \to 0 \quad as \quad n \to \infty$

Theorem

let $z_n = x_n + iy_n \quad , \; n = 1,2, \dots..$ and $S = x + iy$

then $\sum_{n=1}^{\infty} z_n = S \quad iff \quad \sum_{n=1}^{\infty} x_n = x \quad \& \quad \sum_{n=1}^{\infty} y_n = y$

Proof

H.W

Geometrical series

the series $\sum_{n=1}^{\infty} az^{n-1}$ is called geometrical series if

$$\frac{f(n)}{f(n-1)} = constant\ term$$

$\sum_{n=1}^{\infty} az^{n-1} = a + az + az^2 + \cdots$

a is called 1st term in sequence

z is called the base of the sequence

Theorem

The finite sum of geometrical series is $S_n = \frac{a(1-z^n)}{1-z}$ and the infinite sum is $S = \frac{a}{1-z}$

proof

The sum and convergent for Geom. Series .

$$S_n = \sum_{m=1}^{n} az^{m-1}$$

$S_n = a + az + az^2 + \cdots + az^{n-1}$

$zS_n = \quad az + az^2 + \cdots + az^{n-1} + az^n$

$S_n - zS_n = a - az^n$

$S_n(1-z) = a(1-z^n)$

$$\boxed{S_n = \frac{a(1-z^n)}{(1-z)}} \quad \text{partial sum(finite)}$$

The geometrical series convergent when $|z| < 1$ and divergent when $|z| \geq 1$ and

when $|z| < 1$ then $z^n \to 0$ as $n \to \infty$

$$\boxed{S = \frac{a}{(1-z)}} \quad \text{when } |z| < 1 \text{ (infinite sum)}$$

Example

Determine whether the given geometrical series is convergent or divergent, if convergent find the sum.

1. $\sum_{n=1}^{\infty} \frac{3i}{2^n}$

$$\sum_{n=1}^{\infty} \frac{3i}{2^n} = \sum_{n=1}^{\infty} \frac{3i}{2}\left(\frac{1}{2}\right)^{n-1} \Rightarrow \left| \begin{array}{l} a = \frac{3i}{2} \\ z = \frac{1}{2} < 1 \end{array} \right. \text{ convergent}$$

$$S = \frac{a}{1-z} = \frac{3i/2}{1-\frac{1}{2}} = 3i$$

Another method

$$\sum_{n=1}^{\infty} \frac{3i}{2^n} = \frac{3i}{2} + \frac{3i}{4} + \frac{3i}{8} + \cdots$$

$$a = \frac{3i}{2} \quad , z = \frac{f(n)}{f(n-1)} = \frac{\frac{3i}{4}}{\frac{3i}{2}} = \frac{1}{2} < 1 \quad \text{convergent}$$

$$S = \frac{\frac{3i}{2}}{1 - 1/2} = 3i$$

2. $\sum_{n=1}^{\infty} \frac{2i}{3^n}$

$$\sum_{n=1}^{\infty} \frac{2i}{3^n} = \sum_{n=1}^{\infty} \frac{2i}{3}\left(\frac{1}{3}\right)^{n-1} \Rightarrow \begin{cases} a = \frac{2i}{3} \\ z = \frac{1}{3} < 1 \end{cases} \quad \text{convergent}$$

$$S = \frac{2i/3}{1-\frac{1}{3}} = i$$

3. $\sum_{n=1}^{\infty} \frac{(1+2i)^n}{5^n}$

$$= \sum_{n=1}^{\infty} \left(\frac{1+2i}{5}\right)^n = \sum_{n=1}^{\infty} \left(\frac{1+2i}{5}\right)\left(\frac{1+2i}{5}\right)^{n-1}$$

$$a = \frac{1+2i}{5} \qquad , \qquad z = \frac{1+2i}{5}$$

$$|z| = \left|\frac{1}{5} + \frac{2}{5}i\right| = \sqrt{\frac{1}{25} + \frac{4}{25}} = \frac{1}{5} < 1 \quad \text{convergent}$$

$$S = \frac{a}{1-z} = \frac{\frac{1+2i}{5}}{1-\frac{1+2i}{5}} = \frac{1}{2}i$$

4. $\sum_{k=0}^{\infty} (1-i)^k$

$$= \sum_{k=0}^{\infty} (1-\mathrm{i})^{\mathrm{k}} = 1 + (1-i) + (1-i)^2 + \cdots$$

$$a = 1 \qquad , \qquad z = 1 - i$$

$$|z| = |1-i| = \sqrt{2} > 1$$

since $\sqrt{2} > 1$ then the series is divergent and Don't have sum .

5. $\sum_{k=1}^{\infty} \left(\frac{i}{2}\right)^{k}$

$$\sum_{k=1}^{\infty} \left(\frac{i}{2}\right)^{k} = \frac{i}{2} + \left(\frac{i}{2}\right)^{2} + \left(\frac{i}{2}\right)^{3} + \cdots$$

$$a = \frac{i}{2} \qquad , \qquad z = \frac{i}{2}$$

$$|z| = \left|\frac{i}{2}\right| = \frac{1}{2} < 1 \ \textbf{\textit{convergent}}$$

$$S = \frac{a}{1-z} = \frac{\frac{i}{2}}{1-\frac{i}{2}} = \frac{i}{2-i} = \frac{-1}{5} + \frac{2}{5}i$$

6. $\sum_{k=0}^{\infty} 3\left(\frac{2}{1+2i}\right)^{k}$

$$\sum_{k=0}^{\infty} 3\left(\frac{2}{1+2i}\right)^{k} = \sum_{k=0}^{\infty} 3\left(\frac{2}{1+2i}\right)\left(\frac{2}{1+2i}\right)^{k-1}$$

$$a = \frac{6}{1+2i} \qquad , \qquad z = \frac{2}{1+2i}$$

$$|z| = \left|\frac{2}{1+2i}\right| = \frac{2}{\sqrt{5}} < 1 \qquad \textbf{\textit{convergent}}$$

$$S = \frac{a}{1-z} = \frac{\frac{6}{1+2i}}{1-\frac{2}{1+2i}} = \frac{6}{1+2i-2}$$

$$S = \frac{6}{-1+2i} = \frac{-6-12i}{5} = \frac{-6}{5} - \frac{12}{5}i$$

Test of convergent

P-Series

The series $\sum_{k=1}^{\infty}\frac{1}{k^p}$ convergent if $P>1$ and divergent if $P\leq 1$

Example

the series $\sum_{k=1}^{\infty}\frac{i^k}{k^2}$ is absolutely convergent

solution

$$\sum\left|\frac{i^k}{k^2}\right|=\sum\frac{|i^k|}{k^2}=\sum\frac{1}{k^2}$$

By P-series ; P=2

so its absolutely convergent

convergent ⇒ Absolutely convergent , but The converse not true .

Example

is the series $\sum_{k=1}^{\infty}\frac{3}{\sqrt{k}}$ convergent ?

solution

$\sum_{k=1}^{\infty}\frac{3}{\sqrt{k}}=\sum_{k=1}^{\infty}\frac{3}{k^{1/2}}$

the series is divergent by p-test since $p=\frac{1}{2}\leq 1$

Example

show that the series Z_n is convergent

$$Z_n = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots +$$

solution

we can re-write the series Z_n by the form :

$$Z_n = \frac{1}{\sqrt[3]{1}} + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots = \sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{3}}}$$

by p-test , the series is divergent because $p = \frac{1}{3} \leq 1$

Test of convergent

Comparison test

Suppose that $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ be two series which have non-negative terms

if $\sum_{n=1}^{\infty} b_n < \sum_{n=1}^{\infty} a_n$

- if $\sum_{n=1}^{\infty} a_n$ convergent then $\sum_{n=1}^{\infty} b_n$ convergent too
- if $\sum_{n=1}^{\infty} b_n$ divergent then $\sum_{n=1}^{\infty} a_n$ divergent too

Note : To find the second series for the purpose of comparison , The numerator must be a single fixed term or a variable term that can be reduced with the denominator

Example

determine whether the series $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$ convergent or divergent.

solution

We know that $e^{1/n} < e^1$ then the comparison test

by dividing on n^2 and take summation for both sides

$$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2} \leq \sum_{n=1}^{\infty} \frac{e}{n^2} = e \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$a_n = \sum_{n=1}^{\infty} \frac{e}{n^2} \qquad b_n = \sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$$

The series $a_n = \sum_{n=1}^{\infty} \frac{e}{n^2}$ is convergent by P-series with $P = 2$

$\therefore$ the original series $b_n = \sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$ is convergent .

Example

determine whether the series $\sum \frac{n^2-1}{n^4}$ convergent or divergent.

solution

we know $n^2 - 1 < n^2$

by dividing on n^4 and take summation for both sides

$\sum \frac{n^2-1}{n^4} < \sum \frac{n^2}{n^4} = \sum \frac{1}{n^2}$

$b_n = \sum \frac{n^2-1}{n^4} \quad , \quad a_n = \sum \frac{1}{n^2}$

a_n is conv. by P-series with $P = 2$

$\therefore$ the original series $b_n = \sum \frac{n^2-1}{n^4}$ is conv. too

Example

determine whether the series $\sum_{n=1}^{\infty} \frac{n^2}{n^3-3}$ convergent or divergent.

solution

$n^3 - 3 < n^3 \quad \rightarrow \quad \frac{1}{n^3-3} > \frac{1}{n^3}$

by multiply by n^2 and take summation for both sides

$$\underbrace{\sum_{n=1}^{\infty} \frac{n^2}{n^3-3}}_{a_n} > \sum_{n=1}^{\infty} \frac{n^2}{n^3} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{b_n}$$

Since $\sum \frac{1}{n}$ is div. then $\sum_{n=1}^{\infty} \frac{n^2}{n^3-3}$ is also div.

Example

determine whether the series $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ convergent or divergent.

solution

$$(n+1)^2 = n^2 + 2n + 1 > n^2$$

$$(n+1)^2 > n^2$$

$$\underbrace{\frac{1}{(n+1)^2}}_{b_n} < \underbrace{\frac{1}{n^2}}_{a_n}$$

$\because$ $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent so $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ is convergent too

Example

determine whether the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ convergent or divergent.

solution

$$n > \sqrt{n} \rightarrow \frac{1}{n} < \frac{1}{\sqrt{n}}$$

$$\sum_{n=1}^{\infty} \underbrace{\frac{1}{\sqrt{n}}}_{a_n} > \sum_{n=1}^{\infty} \underbrace{\frac{1}{n}}_{b_n}$$

$\because$ $\sum_{n=1}^{\infty} \frac{1}{n}$ is div. by P-series ; $P = 1$ so $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is div. too

Test of convergent

Ratio test (L)

in the series $\sum_{n=1}^{\infty} u_n$

let $L = \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right|$

1. if $L < 1 \Longrightarrow$ the series convergence
2. if $L > 1 \Longrightarrow$ the series divergence
3. if $L = 1 \Longrightarrow$ the series fallen.

Example

determine the interval of convergent for the series $\sum_{n=1}^{\infty} \frac{z^{n+1}}{n}$

Solution

$$u_n = \frac{z^{n+1}}{n} \quad , \quad u_{n+1} = \frac{z^{n+2}}{n+1}$$

since the series is convergent , so $L < 1$

$$L = \lim_{n\to\infty} \left|\frac{z^{n+2}}{n+1} * \frac{n}{z^{n+1}}\right| < 1$$

$$= \lim_{n\to\infty} \frac{n}{n+1} |z| < 1 = |z| < 1$$

The series is convergent when $|z| < 1$