Example

determine the series whether convergent or divergent

$\sum_{n=1}^{\infty} \frac{n+1}{n} * \frac{1}{4^{n-1}}$

Solution

$u_n = \frac{n+1}{n} * \frac{1}{4^{n-1}} \quad , \quad u_{n+1} = \frac{n+2}{n+1} * \frac{1}{4^n}$

$L = lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = lim_{n\to\infty} \left|\frac{n+2}{(n+1)4^n} * \frac{n\,4^{n-1}}{n+1}\right|$

$= \lim_{n\to\infty} \left|\frac{n(n+2)}{4(n+1)^2}\right|$

$= \lim_{n\to\infty} \left|\frac{n^2+2n}{4n^2+8n+4}\right| = \frac{1}{4} < 1$

Then its convergent.

Example

test the conv. of $\sum_{n=1}^{\infty} \frac{1}{n!}$

Solution

$u_n = \frac{1}{n!} \quad , \quad u_{n+1} = \frac{1}{(n+1)!}$

$L = lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = lim_{n\to\infty} \left|\frac{n!}{(n+1)!}\right|$

$= \lim_{n\to\infty} \left|\frac{1}{n+1}\right| = 0 < 1 \quad \boldsymbol{conv.}$

Example

is the series $\sum \frac{z^n}{n^2}$ conv. at $z = 2$

Solution

$$u_n = \frac{2^n}{n^2} \quad , \quad u_{n+1} = \frac{2^{n+1}}{(n+1)^2}$$

$$L = lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = lim_{n\to\infty} \left|\frac{2^{n+1}}{(n+1)^2} * \frac{n^2}{2^n}\right|$$

$$= 2\, lim_{n\to\infty} \left|\frac{n^2}{n^2+2n+1}\right| = 2(1) = 2 > 1$$

$\therefore$ the series is $divergent$.

Example

test the conv. of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!}$

Solution

$$u_n = \frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!} \quad , \quad u_{n+1} = \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$L = lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = lim_{n\to\infty} \left|\frac{(-1)^n z^{2n+1}}{(2n+1)!} * \frac{(2n-1)!}{(-1)^{n-1} z^{2n-1}}\right|$$

$$= lim_{n\to\infty} \left|\frac{z^2}{2n(2n+1)}\right| = 0 < 1 \quad \boldsymbol{conv.}$$

Example

test the conv. of $\sum_{n=1}^{\infty} n!\ z^n$

Solution

$u_n = n!\ z^n \qquad , \qquad u_{n+1} = (n+1)!\ z^{n+1}$

$L = lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = lim_{n\to\infty} \left|\frac{(n+1)!\ z^{n+1}}{n!\ z^n}\right|$

$= \lim_{n\to\infty}(n+1)z = \infty$

$\therefore\ divergent$

Example

test the conv. of $\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n (n^2+1)^{5/2}}$

Solution

$u_n = \frac{(-1)^n n}{4^n (n^2+1)^{5/2}} \qquad , \qquad u_{n+1} = \frac{(n+1)(-1)^{n+1}}{4^{n+1} ((n+1)^2+1)^{5/2}}$

$L = lim_{n\to\infty} \left|\frac{(n+1)(-1)^{n+1}}{4^{n+1} (n^2+2n+2)^{5/2}} * \frac{4^n (n^2+1)^{5/2}}{(-1)^n n}\right|$

$= lim_{n\to\infty} \left|\frac{-(n+1)(n^2+1)^{5/2}}{4n (n^2+2n+1)^{5/2}}\right| = \frac{1}{4} < 1$

$\therefore convergent$

Test of convergent

Root Test (Cauchy Test)

If $\sum_{n=1}^{\infty} a_n$ is a series with positive terms , then the series is convergent if

$$\lim_{n\to\infty}(a_n)^{1/n} < 1$$

and divergent if $(a_n)^{1/n} > 1$

And the test fallen when $(a_n)^{1/n} = 1$

Example

Test the convergence of $\sum_{n=1}^{\infty} \frac{1}{(\log n)^n}$

Solution

$$a_n = \frac{1}{(\log n)^n} \Longrightarrow (a_n)^{1/n} = \left(\frac{1}{(\log n)^n}\right)^{\frac{1}{n}} = \frac{1}{\log n}$$

$$\lim_{n\to\infty} \frac{1}{\log n} = 0 < 1$$

the series $\sum_{n=1}^{\infty} \frac{1}{(\log n)^n}$ is convergence

Example

Test the convergence of $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

Solution

$$a_n = \left(\frac{n}{n+1}\right)^{n^2} \Longrightarrow (a_n)^{1/n} = \left(\left(\frac{n}{n+1}\right)^{n^2}\right)^{\frac{1}{n}} = \left(\frac{n}{n+1}\right)^{n}$$

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n} = \frac{1}{e} < 1$$

the series $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ is convergence

The Power series

The power series have the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

Where $a_0, a_1, \dots, a_n, \dots$ are complex numbers and z may be complex number in region D

Power series properties

1. if $z = z_0$, then the power series convergent to a_0
2. if $z_0 = 0$, then the power series becomes

   $$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

3. the power absolutely convergent if

   $$\sum_{n=0}^{\infty} |a_n (z - z_0)^n| \le |a_0| + |a_1(z - z_0)| + \cdots$$

   are convergent

The Region of convergent (circle convergent)

to know the convergent or divergent to the power series we will use the root test

$\lim_{n\to\infty}|a_n(z-z_0)^n|^{1/n}$

$$= \lim_{n\to\infty}|a_n|^{1/n}|z-z_0| \qquad \ldots\ldots\ldots\ldots..(1)$$

Let $|a_n|^{1/n} = \frac{1}{R}$

eq.(1) be convergent by root test when

$\frac{1}{R}|z-z_0| < 1 \;\; \rightarrow \;\; |z-z_0| < R$

$\therefore$ the series is **convergent** when $|z-z_0| < R$

and **divergent** when $|z-z_0| > R$

The circle of convergent is

$|z-z_0| < R$

$-R < z-z_0 < R$

$z_0 - R < z < z_0 + R$

such that R is the radius of convergence

Example

explain the convergent to the power series $\sum_{n=0}^{\infty} 3^n(z-i)^n$

Solution

$z_0 = i \qquad a_n = 3^n$

To find R , let $\frac{1}{R} = \lim_{n\to\infty}|a_n|^{1/n}$

$\frac{1}{R} = \lim_{n\to\infty}|3^n|^{1/n} \rightarrow \frac{1}{R} = 3 \rightarrow R = \frac{1}{3}$

Then power series is convergent if $|z-i| < \frac{1}{3}$

and divergent if $|z-i| > \frac{1}{3}$

Example

explain the convergent to the power series $\sum_{n=1}^{\infty}\left(\frac{n+1}{n}\right)^n (z-1)^n$

Solution

$z_0 = 1 \quad , \quad a_n = \left(\frac{n+1}{n}\right)^n$

let $\frac{1}{R} = \lim_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^n\right|^{1/n}$

$\frac{1}{R} = \lim_{n\to\infty}\frac{n+1}{n} = 1 \rightarrow R = 1$

Then power series is convergent if $|z-1| < 1$ and divergent if $|z-1| > 1$

Example

explain the convergent to the power series $\sum_{n=1}^{\infty}\left(\frac{n+1}{n}\right)^{n^2}(z-1)^n$

Solution

$z_0=1 \quad , \quad a_n=\left(\frac{n+1}{n}\right)^{n^2}$

let $\frac{1}{R}=\lim_{n\to\infty}|a_n|^{1/n}=\lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^{n^2}\right|^{1/n}$

$\frac{1}{R}=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{n}=e \;\rightarrow R=e^{-1}$

$\therefore$ The power series is convergent if $|z-1|<e^{-1}$

Example

explain the convergent to the power series $\sum_{n=0}^{\infty}\frac{z^{n-1}}{4^{n+1}}$, then find the value of convergent circle , also find the sum of the series .

Solution

$z_0=0 \quad , \quad a_n=\frac{1}{4^{n+1}}$

$\frac{1}{R}=\lim_{n\to\infty}|a_n|^{1/n}$

$\frac{1}{R}=\lim_{n\to\infty}\left|\frac{1}{4^{n+1}}\right|^{1/n}=\lim_{n\to\infty}\left|\frac{1}{4^{1+1/n}}\right|$

$\frac{1}{R}=\frac{1}{4} \;\rightarrow\; R=4$

$\therefore$ the series is conv. when $|z| < 4$ and divergent when $|z| > 4$

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{4^{n+1}} = \frac{z^{-1}}{4} + \frac{1}{4^2} + \frac{z}{4^3} + \frac{z^2}{4^4} + \cdots$$

$$a = \frac{z^{-1}}{4} \quad , \quad base = \frac{\frac{z}{4^3}}{\frac{1}{4^2}} = \frac{z}{4}$$

$$S = \frac{a}{1-z} = \frac{\frac{z^{-1}}{4}}{1-\frac{z}{4}} = \frac{1}{z(4-z)}$$

Example

use Ratio test find the circle of convergent to $\sum_{n=1}^{\infty} \frac{(z+2)^{n-1}}{(n+1)^3\, 4^n}$

Solution

$z_0 = -2$

$$a_n = \frac{1}{(n+1)^3\, 4^n} \quad , \quad a_{n+1} = \frac{1}{(n+2)^3\, 4^{n+1}}$$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^3\, 4^n}{(n+2)^3\, 4^{n+1}}\right|$$

$$L = \frac{1}{4} < 1$$

$\therefore$ the series is convergent

$\because \frac{1}{R} = \frac{1}{4} \to R = 4$

$\therefore$ the series is conv. is

$|z - 2| < 4$

$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{1}{(n+1)^3\, 4^n}\right|^{1/n} = \lim_{n\to\infty} \left|\frac{1}{(n+1)^{3/n}\, 4}\right|$

$$\frac{1}{R} = \frac{1}{4} \;\to R = 4$$

Example

Find the circle and radius of convergent $\sum_{n=0}^{\infty} \frac{1}{(1-2i)^{n+1}} (z-2i)^n$

Solution

$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{1}{(1-2i)^{n+1}}\right|^{1/n}$

$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{1}{(1-2i)^{1+\frac{1}{n}}}\right|$

$\frac{1}{R} = \left|\frac{1}{1-2i}\right| = \frac{1}{\sqrt{5}}$

$R = \sqrt{5}$

The series is convergent if

$$|z-2i| < 5$$

and divergent if

$$|z-2i| > 5$$

Example

Find the circle and radius of convergent $\sum_{n=0}^{\infty}(1+3i)^n\,(z-i)^n$

Solution

$z_0 = i \quad , \quad a_n = (1+3i)^n \;\rightarrow\; \frac{1}{R} = \lim_{n\to\infty}|(1+3i)^n|^{1/n}$

$\frac{1}{R} = |1+3i| = \sqrt{10} \;\rightarrow\; R = \frac{1}{\sqrt{10}}$

∴ the series is convergent when

$$|z-i| < \frac{1}{\sqrt{10}}$$

Example

Find the circle and radius of convergent $\sum_{n=1}^{\infty}\left(\frac{n+1}{n}\right)^n (z-1)^n$

Solution

$z_0 = 1 \quad , \quad a_n = \left(\frac{n+1}{n}\right)^n \;\rightarrow\; \frac{1}{R} = \lim_{n\to\infty}|a_n|^{1/n}$

$\frac{1}{R} = \lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^n\right|^{1/n} = \lim_{n\to\infty}\frac{n+1}{n} = 1$

$$R = 1$$

∴ The series is conv. if $|z-1| < 1$

Zeros of analytic function

If a function $f(z)$ is analytic at z_0 then there exist a circle about z_0 such that $f(z)$ is represented by Taylor series as :

$$f(z) = a_0 + \sum_{n=1}^{\infty} a_0 (z - z_0)^n \quad , \quad |z - z_0| < r$$

Where $a_0 = f(z)$, $a_n = \frac{f^{(n)}(z_0)}{n!}$

if z_0 is a zero of $f(z)$ then $a_0 = 0$ in addition if

$$\boldsymbol{f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0}$$

and $\boldsymbol{f^{(m)}(z_0) \neq 0}$

Note :-

A zero of order one is called simple zero

Example

Find the zeros order of $f(z) = (z-5)^3$ at $z = 5$

Solution

$f(5) = 0$, $f'(z) = 3(z-5)^2 \rightarrow f'(5) = 0$

$f''(z) = 6(z-5) \rightarrow f''(5) = 0$

$f'''(z) = 6 \rightarrow f'''(5) = 6 \neq 0$

Then $f(z)$ has a zero of $\boldsymbol{3rd}$ **order**

Example

Find the zeros order of $\boldsymbol{f(z) = (z-1)^2}$, $\boldsymbol{z = 1}$

Solution

$f(1) = 0$, $f'(z) = 2(z-1) \;\rightarrow\; f'(1) = 0$

$f''(z) = 2 \;\rightarrow\; f''(1) = 2 \neq 0$

$\therefore f(z)$ has a zero of $2nd$ order

Example

Find the zeros order of $\boldsymbol{f(z) = z\, sin(z^2)}$, $\boldsymbol{z_0 = 0}$

Solution

$f(z) = z\, sin(z^2) \rightarrow \; f(0) = 0$

$f'(z) = \sin(z^2) + 2z^2\cos(z^2) \;\rightarrow\; f'(0) = 0$

$f''(z) = -4z^3\cos z^2 + 4z\cos z^2 + 2z\cos z^2 \;\rightarrow f''(0) = 0$

$f'''(z) = -8z^4\cos z^2 - 12z^2\cos z^2 - 8z^2\sin z^2 + 4\cos z^2 - 4z^2\sin z^2 +$
$2\cos z^2$

$f'''(0) = 6 \neq 0$

$\therefore f(z)$ has a zero of $\boldsymbol{3rd}$ **order**

Taylor & maclaurin series

If the function $f(z)$ analytic in the circle that is center is z_0 , then for every point in the circle c_0 can be write at a series

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \cdots$$

When $z_0 = 0$ the series called maclaurin

$$f(z) = f(0) + \frac{f'(0)\,z}{1!} + \frac{f''(0)\,z^2}{2!} + \cdots + \frac{f^{(n)}(0)\,z^n}{n!} + \cdots$$

Some maclaurin series

- $\log(1+z) = z - \frac{z^2}{2!} + \frac{z^3}{3!} - \frac{z^4}{4!} + \cdots = (-1)^{n+1}\sum_{n=1}^{\infty}\frac{z^n}{n!}$
- $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty}\frac{z^n}{n!}$
- $e^{-z} = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{z^n}{n!}$
- $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}$
- $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}$
- $\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots = \sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}$
- $\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots = \sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}$
- $\frac{1}{1-z} = 1 + z + z^2 + \cdots = \sum_{n=1}^{\infty} z^n \quad , \quad |z| < 1$
- $\frac{1}{1+z} = 1 - z + z^2 - \cdots = \sum_{n=1}^{\infty}(-1)^n z^n \quad , \quad |z| < 1$

Note:-

to find taylor series from mac. Series we put $(z - z_0)$ instead all z in series.

Example

find taylor series to $f(z) = \sin z$ at $z_0 = \frac{\pi}{4}$

Solution

We know that mac. series for $\sin z$

$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$

We put $\left(z - \frac{\pi}{4}\right)$ instead of z

$$\sin z = \left(z - \frac{\pi}{4}\right) - \frac{\left(z - \frac{\pi}{4}\right)^3}{3!} + \frac{\left(z - \frac{\pi}{4}\right)^5}{5!} - \cdots$$

Example

find taylor series to e^{-z^2} at $z_0 = 1$

Solution

We know that mac. series for e^{-z^2}

$e^{-z} = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots$

$e^{-z^2} = 1 - \frac{z^2}{1!} + \frac{z^4}{2!} - \frac{z^6}{6!} + \cdots$

We put $(z-1)$ instead of z in series

$$e^{-z^2} = 1 - \frac{(z-1)^2}{1!} + \frac{(z-1)^4}{2!} - \frac{(z-1)^6}{6!} + \cdots$$