Ministry of Higher Education and Scientific Research

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Fourth Stage

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COMPLEX ANALYSIS

1st Course

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CHAPTER TWO - ANALYTIC FUNCTIONS

Some Definitions

(Single value function)

the function $w = f(z)$ is said to be single value function in the Region D if $\forall Z \in D$, there are correspond one value only from values of $f(z)$ such as

$w = z^2 + z + 1$ is single value fun.

(multi-value function)

the function $w = f(z)$ is called multi-value function in Region D if $\forall Z \in D$ correspond many values from Z such that $w = f(z)$ such as $w = \sqrt[3]{z}$ this function for any z will give 3 values to w.

(one-to-one function)

let $w = f(z)$ is a function with Domain D the function f is called one to one(1-1)

iff if $\forall z_1, z_2 \in D$ then $f(z_1) = f(z_2)$

$f(z_1) = f(z_2) \to z_1 = z_2$

Inverse function.

If the function f is (1-1) with Domain D and Range S then the inverse function to f is denoted by f^{-1} is the function with domain is S and Range is D ,

$\forall w \in S, \exists z \in D$ such that $w = f(z)$

to compute the inverse function f^{-1} find z by w then rewrite by formula

$w = f^{-1}(z), \forall z \in S$

Example :

Find Inverse function to $f(z) = \frac{1}{z+2}$, $z \neq -2$

Solution

let w = f(z) , we will write $z = f^{-1}(w)$

$w = \frac{1}{z+2} \rightarrow wz + 2w = 1$

$wz = 1 - 2w \rightarrow z = \frac{1-2w}{w}$

$\therefore f^{-1}(z) = \frac{1-2z}{z}$, $z \neq 0$

(Compose of Function)

let f is function defined over the region D , g is function defined over the region E let for all z ∈ D ,then f(z) ∈ E ,that means all values of f lies in E (domain of g), then we can define the function g(f(z)) on region D, the function gof (Z) called the compose function to f, g

Example :

if $f(z) = 3z + i$, $g(z) = z^2 + z + 1 - i$ **, then find :**

$gof(z)$, $fog\ (z)$

Solution

$gof(z) = g\big(f(z)\big) = g(3z + i)$

$= (3z + i)^2 + (3z + i) + 1 - i$

$= 9z^2 + 6zi - 1 + 3z + i + 1 - i$

$= 9z^2 + 6zi - 3z$

$$fog\,(z) = f\big(g(z)\big) = 3\,(z^2 + z + 1 - i) + i$$

$$= 3z^2 + 3z + 3 - 2i$$

Example :

if $f(z) = z^2 - 1\,,\ g(z) = 2z - 5zi + 1$, then find :

$f(-2+i)\,,f(1-3i)\,,g(2i-3)\,,fog(3i+1)\,,gof(4-i)$

Solution

$$f(-2+i) = (-2+i)^2 - 1$$

$$= 4 - 4i - 1 - 1 = 2 - 4i$$

$$f(1-3i) = (1-3i)^2 - 1$$

$$= 1 - 6i - 9 - 1$$

$$= -9 - 6i$$

$g(2i-3)$

H.W

$fog(3i+1)$

H.W

$gof(4-i)$

H.W

Example :

Write the functions by the form $u(x,y)+iv(x,y)$

Solution

$f(z)=z^3+z+1$

Let $z=x+yi$

$$f(z)=z^3+z+1$$
$$=(x+yi)^3+(x+yi)+1$$
$$=x^3+3x^2yi+3x(yi)^2+(yi)^3+x+yi+1$$
$$=x^3+3x^2yi-3xy^2-y^3i+x+yi+1$$
$$=\underbrace{(x^3-3xy^2+x+1)}_{u}+i\,\underbrace{(3x^2y-y^3+y)}_{v}$$

$f(z)=\frac{z-1}{z+1},\ z\neq -1$

let $z=\ x+yi$

$$f(z)=\frac{z-1}{z+1}$$
$$=\frac{x+yi-1}{x+yi+1}=\frac{x-1+yi}{x+1+yi}\times\frac{x+1-yi}{x+1-yi}$$
$$=\frac{(x^2-1)-(x-1)yi+(x+1)yi+y^2}{(x+1)^2+y^2}$$
$$=\frac{x^2-1+y^2-xyi+yi+xyi+yi}{(x+1)^2+y^2}$$
$$=\frac{(x^2+y^2-1)+2yi}{(x+1)^2+y^2}\rightarrow=\underbrace{\frac{(x^2+y^2-1)}{(x+1)^2+y^2}}_{u}+i\,\underbrace{\frac{2y}{(x+1)^2+y^2}}_{v}$$

Example :

Find the domain of the following functions in complex plane

Solution

$f(z) = \frac{1}{z}$

$D_f = \{\mathbb{C} - \{0\}\}$

$f(z) = \frac{z-1}{z+1}$

$D_f = \{\mathbb{C} - \{-1\}\}$

$f(z) = \frac{1}{z^2+1}$

$z^2 + 1 = 0 \to z = \mp i$

$D_f = \{\mathbb{C} - \{\mp i\}\}$

$f(z) = \frac{1}{1-|z|^2}$

$1 - |z|^2 = 0 \to |z|^2 = 1 \to |z| = 1$

$D_f = \{\mathbb{C} - \{|z| = 1\}\}$

$f(z) = \frac{z}{z+\bar{z}}$

$z + \bar{z} = 2x = 2Re(z)$

$\mathbf{D_f = \{\not{C} - z : \{Re(z) = 0\}\}}$

Example :

Find the Inverse functions in complex plane and check the result

Solution

I. $\boldsymbol{f(z) = \frac{z+1}{z-1}}$

let $w = f(z)$ then $z = f^{-1}(w)$

$$w = \frac{z+1}{z-1}$$

$$wz - w = z + 1$$

$$wz - z = w + 1$$

$$z(w-1) = w + 1 \quad \rightarrow \quad z = \frac{w+1}{w-1}$$

$\therefore f^{-1}(z) = \frac{z+1}{z-1}$

to check the result we will verify $(fof^{-1})(z) = z$

$$fof^{-1}(z) = f\left(\frac{z+1}{z-1}\right)$$

$$= \frac{\frac{z+1}{z-1}+1}{\frac{z+1}{z-1}-1} = \frac{\frac{z+1+z-1}{z-1}}{\frac{z+1-z+1}{z-1}}$$

$$= \frac{2z}{2} = z$$

$\therefore$ the solution is true .

II. $\boldsymbol{f(z) = \sqrt{z-1}}$

let $w = f(z)$ then $z = f^{-1}(w)$

$w = \sqrt{z-1}$

$w^2 = z - 1 \rightarrow z = w^2 + 1$

$f^{-1}(z) = z^2 + 1$

to check the result we will verify $(fof^{-1})(z) = z$

$fof^{-1}(z) = f(z^2 + 1)$

$= \sqrt{z^2 + 1 - 1} = \sqrt{z^2} = z$

III. $\boldsymbol{f(z) = \frac{2z-1}{z+2}}$

H.W

(Metric Space in complex no.)

If $d: \mathbb{C} \times \mathbb{C} \to R^+$ is mapping then the $d(z_1, z_2)$ is a metric space if the following three conditions are satisfies :

1. $d(z_1, z_2) \geq 0$
2. $d(z_1, z_2) = d(z_2, z_1)$
3. $d(z_1, z_3) \leq d(z_1, z_2) + d(z_2, z_3)$

We can define the metric in complex variables as

$d(z_1, z_2) = |z_1 - z_2|$

Example :

prove that the complex variables is a metric space

Solution

define $d: \mathbb{C} \times \mathbb{C} \to R^+$

let $z_1, z_2, z_3 \in \mathbb{C}$

$d(z_1, z_2) = |z_1 - z_2|$

$M_1)$ $d(z_1, z_2) = |z_1 - z_2| \geq 0$

$M_2)$ $d(z_1, z_2) = |z_1 - z_2| = |z_2 - z_1| = d(z_2, z_1)$

$M_3)$ $d(z_1, z_3) = |z_1 - z_3|$

$= |z_1 - z_2 + z_2 - z_3|$

$\leq |z_1 - z_2| + |z_2 - z_3|$

$\leq d(z_1, z_2) + d(z_2, z_3)$

$\therefore \quad d(z_1, z_3) \leq d(z_1, z_2) + d(z_2, z_3)$

$\therefore$ metric space

The Limits

for any $+ive$ no. ε $\exists\delta > 0$

$$|f(z) - w_0| < \varepsilon \text{ if } |z - z_0| < \delta$$

Such that w_0 is the value of limit and z_0 the value of approaching and written as $\lim_{z\to z_0} f(z) = w_0$ this method called the Definition

Example :

prove $\lim_{z\to 2i-1}(2z+3) = 4i+1$

Solution

Let $\varepsilon > 0$, then we are prove that

$|f(z) - w_0| < \varepsilon$ if $|z - z_0| < \delta$, such that $z_0 = 2i - 1$ and $w_0 = 4i + 1$

$|2z + 3 - (4i + 1)| < \varepsilon$ when $|z - (2i - 1) <| \delta$

$|2z + 3 - 4i - 1| < \varepsilon$ when $|z - 2i + 1| < \delta$

$|2z - 4i + 2| < \varepsilon$ when $|z - 2i + 1| < \delta$

$|2(z - 2i + 1)| < \varepsilon$ when $|z - 2i + 1| < \delta$

$|z - 2i + 1| < \frac{\varepsilon}{2}$ when $|z - 2i + 1| < \delta$

$\delta = \frac{\varepsilon}{2}$

$\therefore$The limit is true