Example :

prove $\lim_{z\to1}\frac{z^2-1}{z-1}=2$

Solution

Let $\varepsilon>0$, then we are prove that

$|f(z)-w_0|<\varepsilon$ when $|z-z_0|<\delta$, such that $z_0=1$ and $w_0=2$

$\left|\frac{z^2-1}{z-1}-2\right|<\varepsilon$ when $|z-1|<\delta$

$\left|\frac{(z-1)(z+1)}{z-1}-2\right|<\varepsilon$ when $|z-1|<\delta$

$|z+1-2|<\varepsilon$ when $|z-1|<\delta$

$|z-1|<\varepsilon$ when $|z-1|<\delta$

$\varepsilon=\delta$

$\therefore$ the limit is true

Example :

prove $\lim_{z\to2}5z-3=7$

Solution

Let $\varepsilon>0$, then we are prove that

$|f(z)-w_0|<\varepsilon$ when $|z-z_0|<\delta$, such that $z_0=2$ and $w_0=7$

$|5z-3-7|<\varepsilon$ when $|z-2|<\delta$

$|5z-10|<\varepsilon$ when $|z-2|<\delta$

$|5(z-2)|<\varepsilon$ when $|z-2|<\delta$

$|z-2|<\frac{\varepsilon}{5}$ when $|z-2|<\delta$

$\delta = \frac{\varepsilon}{5}$

$\therefore$ the limit is satisfy

Example :

prove $\lim_{z\to 3} \frac{4z^2-36}{z-3} = 24$

Solution

Let $\varepsilon > 0$, then we are prove that

$|f(z) - w_0| < \varepsilon$ when $|z - z_0| < \delta$, such that $z_0 = 3$ and $w_0 = 24$

$\left|\frac{4z^2-36}{z-3} - 24\right| < \varepsilon$ when $|z-3| < \delta$

$\left|\frac{4(z-3)(z+3)}{z-3} - 24\right| < \varepsilon$ when $|z-3| < \delta$

$|4z + 12 - 24| < \varepsilon$ when $|z-3| < \delta$

$|4z - 12| < \varepsilon$ when $|z-3| < \delta$

$|z-3| < \frac{\varepsilon}{4}$ when $|z-3| < \delta$

$\delta = \frac{\varepsilon}{4}$

$\therefore$The limit is true

Example :

prove $\lim_{z\to 1} \frac{zi}{2} = \frac{i}{2}$

Solution

Let $\varepsilon > 0$, then we are prove that

$|\boldsymbol{f(z) - w_0}| < \boldsymbol{\varepsilon}$ when $|\boldsymbol{z - z_0}| < \boldsymbol{\delta}$, such that $\boldsymbol{z_0} = 1$ and $\boldsymbol{w_0} = \frac{i}{2}$

$\left|\frac{zi}{2} - \frac{i}{2}\right| < \varepsilon$ when $|z - 1| < \delta$

$\left|\frac{i}{2}(z - 1)\right| < \varepsilon$ when $|z - 1| < \delta$

$\left|\frac{i}{2}\right| |z - 1| < \varepsilon$ when $|z - 1| < \delta$

$\frac{1}{2}|z - 1| < \varepsilon$ when $|z - 1| < \delta$

$|z - 1| < 2\varepsilon$ when $|z - 1| < \delta$

$\delta = 2\varepsilon$

∴The limit is true

Example :

prove $\lim_{z\to 3i} \frac{z^2 5iz-6}{z-3i} = i$

Solution

H.W

Example :

Prove the function $f(z) = \frac{z}{\bar{z}}$ don't have limit at $z = 0$

Solution

Let $z = (x, y)$, When z approaches to zero (x, y) approaches to zero

we take two paths

$z = (x, 0)$ and x approaches to 0 , or $z = (0, y)$ and y approaches to 0

When $z = (x, 0) \to y = 0$

$$\lim_{\underbrace{(x,y)\to(0,y)}_{y=0}} f(z) = \lim_{\underbrace{(x,y)\to(0,y)}_{y=0}} \frac{z}{\bar{z}} = \lim_{\underbrace{(x,y)\to(0,y)}_{y=0}} \frac{x+yi}{x-yi} = \lim_{x\to 0} \frac{x}{x} = \boxed{1}$$

When $z = (0, y) \to x = 0$

$$\lim_{\underbrace{(x,y)\to(x,0)}_{x=0}} f(z) = \lim_{\underbrace{(x,y)\to(x,0)}_{x=0}} \frac{z}{\bar{z}} = \lim_{\underbrace{(x,y)\to(x,0)}_{x=0}} \frac{x+yi}{x-yi} = \lim_{y\to 0} \frac{yi}{-yi} = \boxed{-1}$$

∴ the limit don't exists.

The continuity

the function $f(z)$ is continuous at z_0 if the conditions satisfied

I. $\lim_{z\to z_0} f(z)$ exist
II. $f(z_0)$ exist
III. $\lim_{z\to z_0} f(z) = f(z_0)$

Example :

is the function f continuous at $z = -2i$?

$$f(z) = \begin{cases} \dfrac{z^2+4}{z+2i}, & z \neq -2i \\ -4i, & z = 2i \end{cases}$$

Solution

❶ $\lim_{z\to -2i} \frac{z^2+4}{z+2i} = \lim_{z\to -2i} \frac{(z+2i)(z-2i)}{z+2i}$

$\lim_{z\to -2i} z - 2i = -2i - 2i = -4i$

❷ $f(-2i) = -4i$

❸ $\lim_{z\to -2i} f(z) = f(z_0) = -4i$

∴the function is continuous at $\boldsymbol{z = -2i}$

Example :

is the function f continuous at $z = 3i$?

$$f(z) = \begin{cases} \dfrac{2z^2+18}{z-3i}, & z \neq 3i \\ 12i, & z = 3i \end{cases}$$

Solution

❶ $f(3i) = 12i$

❷ $\lim_{z\to 3i} \frac{2z^2+18}{z-3i} = \lim\limits_{z\to 3i} \frac{2(z^2+9)}{z-3i}$

$\lim_{z\to 3i} \frac{2(z-3i)(z-3i)}{z-3i} = 2(6i) = 12i$

❸ $\lim_{z\to 3i} f(z) = f(z_0) = 12i$

$\therefore$ the function is continuous at $z = 3i$

Example :

is the function f continuous at $z = 3i$?

$$f(z) = \begin{cases} \dfrac{z^2 - iz + 6}{z^2 - 9} & , z \neq 3i \\ \dfrac{5}{6} & , z = 3i \end{cases}$$

Solution

❶ $f(3i) = \frac{5}{6}$

❷ $\lim_{z \to 3i} \frac{z^2 - iz + 6}{z^2 - 9}$

$\lim_{z \to 3i} \frac{(z-3i)(z+2i)}{(z-3i)(z+3i)} = \frac{5i}{6i} = \frac{5}{6}$

❸ $\lim_{z \to 3i} f(z) = f(z_0) = \frac{5}{6}$

$\therefore$ the function is continuous at $z = 3i$

Example :

is the function f continuous at $z = -4i$?

$$f(z) = \begin{cases} \dfrac{z^3 - 64i}{z + 4i} , & z \neq -4i \\ 48 , & z = -4i \end{cases}$$

Solution

❶ $f(-4i) = -64$

❷ $\lim_{z \to -4i} \frac{z^3 - 64i}{z + 4i}$

$$= \lim_{z \to -4i} \frac{z^3+64i^3}{z+4i}$$

$$= \lim_{z \to -4i} \frac{(z+4i)(z^2-4zi-16)}{(z+4i)} = -16-16-16 = -48$$

❸ $\lim_{z \to -4i} f(z) \neq f(-4i)$

$\therefore$ the function is not continuous at $z = -4i$

Example :

is the function f continuous at $z = 2i$?

$$f(z) = \begin{cases} \dfrac{2z^2+8}{z^2-iz+2}, & z \neq 2i \\ 8/3, & z = 2i \end{cases}$$

Solution

H.W

The Uniformly continuity

from the definition of continuity we note that δ depends on ε and z_0 if we can find δ such that δ depends only on ε and don't depends on $z_0 \in D$, then the fun. call it uniformly continuous in D then we can say

$$|f(z_2) - f(z_1)| < \varepsilon \quad \textbf{When} \quad |z_2 - z_1| < \delta$$

Example :

is the function $f(z) = z^2$ uniformly continuous at $|z| < 1$?

Solution

we shall prove for all $\varepsilon > 0$, $\exists\, z_1, z_2 \in D$ such that

$|f(z_2) - f(z_1)| < \varepsilon$ When $|z_2 - z_1| < \delta$

$|{z_2}^2 - {z_1}^2| < \varepsilon$ When $|z_2 - z_1| < \delta$

$= |(z_2 - z_1)(z_2 + z_1)| < \varepsilon$ When $|z_2 - z_1| < \delta$

$= |(z_1 + z_2)||(z_2 - z_1)| < \varepsilon$ When $|z_2 - z_1| < \delta$

$\leq (|z_1| + |z_2|)|(z_2 - z_1)| < \varepsilon$ When $|z_2 - z_1| < \delta$

$\leq (1 + 1)|(z_2 - z_1)| < \varepsilon$ When $|z_2 - z_1| < \delta$

$\leq 2|(z_2 - z_1)| < \varepsilon$ When $|z_2 - z_1| < \delta$

$\leq |(z_2 - z_1)| < \frac{\varepsilon}{2}$ When $|z_2 - z_1| < \delta$

$|f(z_2) - f(z_1)| \leq \frac{\varepsilon}{2}$ When $|z_2 - z_1| < \delta$

$$\delta = \frac{\varepsilon}{2}$$

$\therefore$ Uniformly Cont.

Example :

is the function $f(z) = z^3$ uniformly continuous at $|z| < 2$?

Solution

we shall prove for all $\varepsilon > 0$, $\exists\, z_1, z_2 \in D$ such that

$|f(z_2) - f(z_1)| < \varepsilon$ When $|z_2 - z_1| < \delta$

$|z_2{}^3 - z_1{}^3| < \varepsilon$ When $|z_2 - z_1| < \delta$

$= |(z_2 - z_1)(z_2{}^2 + z_1 z_2 + z_1{}^2)| < \varepsilon$ When $|z_2 - z_1| < \delta$

$= |z_1 - z_2||z_2{}^2 + z_1 z_2 + z_1{}^2| < \varepsilon$ When $|z_2 - z_1| < \delta$

$\leq |(z_2 - z_1|(|z_2|^2 + |z_1||z_2| + |z_1|^2) < \varepsilon$ When $|z_2 - z_1| < \delta$

$\leq |(z_2 - z_1)|(4 + 4 + 4) < \varepsilon$ When $|z_2 - z_1| < \delta$

$\leq 12|(z_2 - z_1)| < \varepsilon$ When $|z_2 - z_1| < \delta$

$\leq |(z_2 - z_1)| < \frac{\varepsilon}{12}$ When $|z_2 - z_1| < \delta$

$|f(z_2) - f(z_1)| \leq \frac{\varepsilon}{12}$ When $|z_2 - z_1| < \delta$

$$\delta = \frac{\varepsilon}{12}$$

$\therefore$ Uniformly Continuous

Example :

is the function $f(z) = \frac{1}{z}$ uniformly continuous at $|z| < 1$?

Solution

we shall prove for all $\varepsilon > 0$, $\exists\, z_1, z_2 \in D$ such that

$|f(z_2) - f(z_1)| < \varepsilon$ When $|z_2 - z_1| < \delta$

$\left|\frac{1}{z_2} - \frac{1}{z_1}\right| < \varepsilon$ When $|z_2 - z_1| < \delta$

$= \left|\frac{z_1 - z_2}{z_1 z_2}\right| < \varepsilon$ When $|z_2 - z_1| < \delta$

$= |z_1 - z_2| > \varepsilon$ When $|z_2 - z_1| < \delta$

$= |z_2 - z_1| > \varepsilon$ When $|z_2 - z_1| < \delta$

we note that no relation between ε , δ

∴ the fun. Is not Uniformly Continuous

Example :

is the function $f(z) = 3z - 2$ uniformly continuous at $|z| < 5$?

Solution

H.W