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complex analysis

2nd Course

Chapter 2

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Chapter Two

Laurent Theorem

Laurent Theorem

Let $w = f(z)$ be analytic function in the ring $r_2 < |z - z_o| < r_1$, then at any point in this ring the function f(z) can be expressed as :

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_o)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_o)^n}$$

Notes :

- We must transform the given function to one of these formulas :

$$\frac{1}{1-u} \quad or \quad \frac{1}{1+u}$$

- $\frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots$
- $\frac{1}{1+u} = 1 - u + u^2 - u^3 + \cdots$
- To transform any function Laurent series , the following two conditions must be satisfied :
- The first element may be " 1 " only no others .
- The function u must satisfy $|u| < 1$

Example 1 :

Find Laurent Series for the function :

$$\mathbf{f(z) = \frac{1}{z^2(1-z)}}$$

In the regions : 1) |z| > 1 , 2) |z| < 1

Solution :

$$f(z) = \frac{1}{z^2(1-z)}$$

$$|z| > 1 \Rightarrow \left|\frac{1}{z}\right| < 1$$

$$f(z) = \frac{1}{z^2(1-z)} = \frac{1}{z^2}\frac{1}{-z(1-\frac{1}{z})} = \frac{-1}{z^3}\frac{1}{1-\frac{1}{z}}$$

$$f(z) = -\left[\frac{1}{z^3} + \frac{1}{z^4} + \frac{1}{z^5} + \cdots\right]$$

$$f(z) = -\sum_{n=1}^{\infty}\frac{1}{z^{n+2}}$$

$$|z| < 1$$

$$f(z) = \frac{1}{z^2(1-z)} = \frac{1}{z^2}\frac{1}{(1-z)} = \frac{1}{z^2}[1 + z + z^2 + z^3 + \cdots]$$

$$f(z) = \left[\frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \cdots\right] = \sum_{n=0}^{\infty} z^{n-2}$$

Example 2 :

Write the function as Laurent Series :

$$\mathbf{f(z) = \frac{z+1}{z-1}} \quad , \quad \mathbf{|z| > 1}$$

Solution :

$$f(z) = \frac{z+1}{z-1}$$

$\frac{1}{z-1}$ *in the region* $|z| > 1$

$$|z| > 1 \implies \left|\frac{1}{z}\right| < 1$$

$$\frac{z+1}{z-1} = \frac{z+1}{z(1-\frac{1}{z})} = \frac{z+1}{z}\frac{1}{1-\frac{1}{z}}$$

$$\frac{z+1}{z-1} = \frac{z+1}{z}\left[1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots\right]$$

$$\frac{z+1}{z-1} = (z+1)\left[\left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^3 + \cdots\right]$$

$$\frac{z+1}{z-1} = \sum_{n=1}^{\infty} \frac{z+1}{z^n}$$

Example 3 :

Find Laurent Series for the function :

$$\mathbf{f(z) = \frac{-1}{(z-1)(z-2)}} \quad , \quad \mathbf{1 < |z| < 2}$$

Solution :

$$f(z)=\frac{-1}{(z-1)(z-2)}$$

$$\frac{-1}{(z-1)(z-2)}=\frac{A}{z-1}+\frac{B}{z-2}$$

$$-1=A(z-2)+B(z-1)$$

$$z=1 \Rightarrow A=1$$

$$z=2 \Rightarrow B=-1$$

$$\frac{-1}{(z-1)(z-2)}=\frac{1}{z-1}-\frac{1}{z-2}$$

$\frac{1}{z-1}$ *in the region* $|z|>1$

$$|z|>1 \Rightarrow \left|\frac{1}{z}\right|<1$$

$$\frac{1}{z-1}=\frac{1}{z(1-\frac{1}{z})}=\frac{1}{z}\frac{1}{1-\frac{1}{z}}$$

$$\frac{1}{z}\frac{1}{1-\frac{1}{z}}=\frac{1}{z}\left[1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\cdots\right]$$

$$\frac{1}{z-1}=\left[\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\left(\frac{1}{z}\right)^3+\cdots\right]=\sum_{n=1}^{\infty}\left(\frac{1}{z}\right)^n$$

$\frac{1}{z-2}$ *in the region* $|z|<2$

$$|z|<2 \Rightarrow \left|\frac{z}{2}\right|<1$$

$$\frac{1}{z-2}=\frac{1}{-2(1-\frac{z}{2})}=-\frac{1}{2}\frac{1}{1-\frac{z}{2}}$$

$$-\frac{1}{2}\frac{1}{1-\frac{z}{2}}=-\frac{1}{2}\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right]$$

$$\frac{1}{z-2}=-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n=-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$

$$f(z)=\frac{-1}{(z-1)(z-2)}=\frac{1}{z-1}-\frac{1}{z-2}$$

$$f(z)=\sum_{n=1}^{\infty}\left(\frac{1}{z}\right)^n+\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$

Example 4 :

Find Laurent Series for the function

$f(z)=\frac{1}{z^3-2z^2}$ in the regions : A) $|z|>2$ B) $|z|<2$

Solution

H.W

Example 5 :

Find Laurent Series for the function

$f(z) = \frac{1}{z^2-5z+4}$ **in the regions : A)** $1 < |z| < 4$ **B)** $|z| < 1$ **C)** $|z| > 4$

Solution

H.W

Example 6 :

Find Laurent Series for the function :

$$\mathbf{f(z) = \frac{1}{(z-2)(z-3)}} \quad , \quad 2 < |z| < 3$$

Solution :

$$f(z) = \frac{1}{(z-2)(z-3)}$$

$$\frac{1}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}$$

$$1 = A(z-3) + B(z-2)$$

$$z = 2 \Longrightarrow A = -1$$

$$z = 3 \Longrightarrow B = 1$$

$$\frac{1}{(z-2)(z-3)} = \frac{-1}{z-2} + \frac{1}{z-3}$$

$\frac{-1}{z-2}$ *in the region* $|z| > 2$

$$|z| > 2 \implies \left|\frac{z}{2}\right| > 1 \implies \left|\frac{2}{z}\right| < 1$$

$$\frac{-1}{z-2} = \frac{1}{z(1-\frac{2}{z})} = \frac{1}{z}\frac{1}{1-\frac{2}{z}}$$

$$f(z) = \frac{-1}{z}\left[1 + \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \cdots\right]$$

$$f(z) = -\left[\frac{1}{z} + \frac{2}{z^2} + \frac{2^2}{z^3} + \frac{2^3}{z^4} + \frac{2^4}{z^5} + \cdots\right] = -\sum_{n=1}^{\infty}\frac{2^{n-1}}{z^n}$$

$\frac{1}{z-3}$ *in the region* $|z| < 3$

$$|z| < 3 \implies \left|\frac{z}{3}\right| < 1$$

$$\frac{1}{z-3} = \frac{1}{-3(1-\frac{z}{3})} = -\frac{1}{3}\frac{1}{1-\frac{z}{3}}$$

$$f(z) = -\frac{1}{3}\frac{1}{1-\frac{z}{3}} = -\frac{1}{3}\left[1 + \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 + \cdots\right]$$

$$f(z) = -\left[\frac{1}{3} + \frac{z}{3^2} + \frac{z^2}{3^3} + \frac{z^3}{3^4} + \cdots\right] = -\sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}}$$

$$f(z) = \frac{1}{(z-2)(z-3)} = \frac{-1}{z-2} + \frac{1}{z-3}$$

$$f(z) = -\sum_{n=1}^{\infty}\frac{2^{n-1}}{z^n} - \sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}}$$