Example 7 :

Find Laurent Series for the function :

$f(z) = \frac{1}{z^2-1}$ **in the regions : A) $|z| > 1$ B) $|z| < 1$**

Solution :

H.W

Example 8 :

Find Laurent Series for the function :

$$f(z) = \frac{z+1}{z(z-3)} \quad , \quad 1 < |z-1| < 2$$

Solution :

$$\frac{z+1}{z(z-3)} = \frac{A}{z} + \frac{B}{z-3}$$

$$z+1 = A(z-3) + Bz$$

$$z = 0 \Rightarrow A = -{}^{1}/_{3}$$

$$z = 3 \Rightarrow B = {}^{4}/_{3}$$

$$\frac{z+1}{z(z-3)} = \frac{-\frac{1}{3}}{z} + \frac{\frac{4}{3}}{z-3}$$

$$\frac{-\frac{1}{3}}{z} = \frac{-1}{3}\frac{1}{z-1+1}$$

$\frac{1}{z-1+1}$ *in the region* $|z-1| > 1$

$$|z-1| > 1 \implies \left|\frac{1}{z-1}\right| < 1$$

$$\frac{-1}{3}\frac{1}{z-1+1} = \frac{-1}{3}\frac{1}{(z-1)(1+\frac{1}{z-1})}$$

$$f(z) = \frac{-1}{3(z-1)}\left[1 - \left(\frac{1}{z-1}\right) + \left(\frac{1}{z-1}\right)^2 - \left(\frac{1}{z-1}\right)^3 + \cdots\right]$$

$$f(z) = \frac{1}{3}\left[-\left(\frac{1}{z-1}\right) + \left(\frac{1}{z-1}\right)^2 - \left(\frac{1}{z-1}\right)^3 + \left(\frac{1}{z-1}\right)^4 + \cdots\right]$$

$$f(z) = \frac{1}{3}\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{z-1}\right)^n$$

$$\frac{\frac{4}{3}}{z-3} = \frac{\frac{4}{3}}{z-1-2}$$

$\frac{\frac{4}{3}}{z-1-2}$ *in the region* $|z-1| < 2$

$$|z-1| < 2 \implies \left|\frac{z-1}{2}\right| < 1$$

$$\frac{\frac{4}{3}}{z-1-2} = \frac{\frac{4}{3}}{-2(1-\frac{z-1}{2})} = -\frac{2}{3}\frac{1}{(1-\frac{z-1}{2})}$$

$$f(z) = -\frac{2}{3}\left[1+\left(\frac{z-1}{2}\right)+\left(\frac{z-1}{2}\right)^2+\left(\frac{z-1}{2}\right)^3+\cdots\right]$$

$$f(z) = -\frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n$$

$$f(z) = \frac{1}{3}\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{z-1}\right)^n - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n$$

Example 9 :

Find Laurent Series for the function :

$$f(z) = \frac{2z-1}{(z+1)(z-2)} \quad , \quad 1 < |z-3| < 4$$

Solution :

H.W

Example 10 :

Find Laurent Series for the function :

$$\mathbf{f(z) = \frac{3}{(z-1)(z+2)}} \quad , \quad \mathbf{1 < |z+1| < 2}$$

Solution :

$$\frac{3}{(z-1)(z+2)} = \frac{A}{z-1} + \frac{B}{z+2}$$

$$3 = A(z+2) + B(z-1)$$

$$z = 1 \Rightarrow A = 1$$

$$z = -2 \Rightarrow B = -1$$

$$\frac{3}{(z-1)(z+2)} = \frac{1}{z-1} - \frac{1}{z+2}$$

$$\frac{1}{z-1} = \frac{1}{z+1-1-1}$$

$\frac{1}{z+1-2}$ *in the region* $|z+1| < 2$

$$|z+1| < 2 \;\Rightarrow \left|\frac{z+1}{2}\right| < 1$$

$$\frac{1}{z+1-2} = \frac{1}{-2(1-\frac{z+1}{2})}$$

$$f(z) = \frac{-1}{2}\left[1 + \left(\frac{z+1}{2}\right) + \left(\frac{z+1}{2}\right)^2 + \left(\frac{z+1}{2}\right)^3 + \cdots\right]$$

$$f(z) = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z+1}{2}\right)^n$$

$$\frac{1}{z+2}=\frac{1}{z+1+1}$$

$\frac{1}{z+1+1}$ *in the region* $|z+1|>1$

$$|z+1|>1 \Rightarrow \left|\frac{1}{z+1}\right|<1$$

$$\frac{1}{z+1-1}=\frac{1}{(z+1)(1+\frac{1}{z+1})}=\frac{1}{z+1}\frac{1}{(1+\frac{1}{z+1})}$$

$$f(z)=\frac{1}{z+1}\left[1-\left(\frac{1}{z+1}\right)+\left(\frac{1}{z+1}\right)^2-\left(\frac{1}{z+1}\right)^3+\cdots\right]$$

$$f(z)=\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{z+1}\right)^n$$

$$f(z)=-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z+1}{2}\right)^n-\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{z+1}\right)^n$$

Example 11 :

Find Laurent Series for the function :

$$f(z)=\frac{z+1}{(z-1)(z-3)} \quad , \quad |z-2|<1$$

Solution :

H.W

Example 12 :

Find Laurent Series for the function :

$$\mathbf{f(z) = \frac{5z+2}{z(z+1)} \quad , \quad 1 < |z-1| < 2}$$

Solution :

$$\frac{5z+2}{z(z+1)} = \frac{A}{z} + \frac{B}{z+1}$$

$$5z + 2 = A(z+1) + Bz$$

$$z = 0 \Longrightarrow A = 2$$

$$z = -1 \Longrightarrow B = 3$$

$$\frac{5z+2}{z(z+1)} = \frac{2}{z} + \frac{3}{z+1}$$

$$\frac{2}{z} = \frac{2}{z-1+1}$$

$\frac{2}{z-1+1}$ *in the region* $|z-1| > 1$

$$|z-1| > 1 \;\Longrightarrow \left|\frac{1}{z-1}\right| < 1$$

$$\frac{2}{z-1+1} = \frac{2}{(z-1)(1+\frac{1}{z-1})}$$

$$f(z) = \frac{2}{z-1}\left[1 - \left(\frac{1}{z-1}\right) + \left(\frac{1}{z-1}\right)^2 - \left(\frac{1}{z-1}\right)^3 + \cdots\right]$$

$$f(z) = 2\left[\left(\frac{1}{z-1}\right) - \left(\frac{1}{z-1}\right)^2 + \left(\frac{1}{z-1}\right)^3 - \left(\frac{1}{z-1}\right)^4 + \cdots\right]$$

$$f(z) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{z-1}\right)^{n}$$

$$\frac{3}{z+1} = \frac{3}{z-1+2}$$

$\frac{3}{z-1+2}$ *in the region* $|z-1| < 2$

$$|z-1| < 2 \Rightarrow \left|\frac{z-1}{2}\right| < 1$$

$$\frac{3}{z-1+2} = \frac{3}{2(1+\frac{z-1}{2})} = \frac{3}{2}\frac{1}{(1+\frac{z-1}{2})}$$

$$f(z) = \frac{3}{2}\left[1-\left(\frac{z-1}{2}\right)+\left(\frac{z-1}{2}\right)^{2}-\left(\frac{z-1}{2}\right)^{3}+\cdots\right]$$

$$f(z) = \frac{3}{2}\sum_{n=0}^{\infty}(-1)^{n}\left(\frac{z-1}{2}\right)^{n}$$

$$f(z) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{z-1}\right)^{n} + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^{n}\left(\frac{z-1}{2}\right)^{n}$$

Example 13 :

Find Laurent Series for the function :

$$f(z) = \frac{z}{(z+3)(z+4)} \quad , \quad 4 < |z-1| < 5$$

Solution :

H.W