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Fourth Stage

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# COMPLEX ANALYSIS

## 1st Course

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## CHAPTER THREE – DIFFERENTION IN C

### The Derivative

Let f be a function Whose domain of definition contain a neighborhood

$|z - z_0| < \varepsilon$ a point $z_0$.

the derivative of f at the point to $z_0$ is

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Let $\Delta z = z - z_0$

When $z \to z_0$ , then $\Delta z \to 0$

then the derivative can be written as:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

the previous formula called the derivative at a point $z_0$ , and the formula

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

the previous formula called the derivative at any point

**Example :**

**By using Definition find $f'(z)$ to $f(z) = z^2$**

**Solution**

$$f'(z) = lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z}$$

$$= lim_{\Delta z\to 0}\frac{z^2+2z\Delta z-\Delta^2 z-z^2}{\Delta z}$$

$$\acute{f}(z) = \lim_{\Delta z\to 0}\frac{\Delta z(2z-\Delta z)}{\Delta z}$$

$$= \lim_{\Delta z\to 0} 2z - \Delta z = 2z - 0 = 2z$$

**Example :**

**By using Definition find $f'(3+i)$ to $f(z) = 2z^2$**

**Solution**

$$f'(z_o) = \lim_{\Delta z\to 0}\frac{f(z_o+\Delta z)-f(z_o)}{\Delta z}$$

$$f'(3+i) = lim_{\Delta z\to 0}\frac{f(3+i+\Delta z)-f(3+i)}{\Delta z}$$

$$= lim_{\Delta z\to 0}\frac{2(3+i+\Delta z)^2-2(3+i)^2}{\Delta z}$$

$$= lim_{\Delta z\to 0}\frac{2(9-1+\Delta^2 z+6i+6\Delta z+2i\Delta z)-2(9+6i-1)}{\Delta z}$$

$$= lim_{\Delta z\to 0}\frac{18-2+2\Delta^2 z+12i+12\Delta z+4i\Delta z-18-12i+2}{\Delta z}$$

$$= lim_{\Delta z\to 0}\frac{\Delta z[2\Delta z+12+4i]}{\Delta z}$$

$$= \lim_{\Delta z\to 0} 2\Delta z + 12 + 4i$$

$$= 12 + 4i$$

**Note:-** $\boxed{(\mathbf{A}\pm\mathbf{B}\pm\mathbf{C})^2 = \mathbf{A}^2+\mathbf{B}^2+\mathbf{C}^2 \pm 2\mathbf{AB} \pm 2\mathbf{AC} \pm 2\mathbf{BC}}$

**Example :**

**By using Definition find $f'(z)$ to $f(z) = 3z^3$**

**Solution**

$$f'(z) = \lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{3(z+\Delta z)^3 - 3z^3}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{3(z^3 + 3z^2\Delta z + 3z\,\Delta^2 z + \Delta^3 z) - 3z^3}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{3z^3 + 9z^2\Delta z + 9z\,\Delta^2 z + 3\Delta^3 z - 3z^3}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{\Delta z(9z^2 + 9z\,\Delta z + 3\Delta^2 z)}{\Delta z}$$

$$= 9z^2 + 9z\,\Delta z + 3\Delta^2 z = 9z^2$$

**Example :**

**By using Definition find $f'(z)$ to $f(z) = \frac{1}{z}$**

**Solution**

$$f'(z) = \lim_{\Delta z\to 0} \frac{\frac{1}{(z+\Delta z)} - \frac{1}{z}}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{\frac{z - z - \Delta z}{z(z+\Delta z)}}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{-\Delta z}{z(z+\Delta z)} \cdot \frac{1}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{-1}{z(z+\Delta z)} = \frac{-1}{z^2}$$

**Example :**

**prove that if $f(z) = \bar{z}$ (such that $Im(z) \neq 0$) them $f'(z)$ Doesn't exists**

**Solution**

let $z = x + yi$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{(\Delta x, \Delta y) \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x - i0}{\Delta x + i0} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

$$f'(z) = \lim_{\substack{\Delta y \to 0 \\ \Delta x = 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{0 - i\Delta y}{0 + i\Delta y} = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$$

The limit at two paths are different

The limit doesn't exist

Them the function don't have any derivative

**Example :**

**By using Definition find $f'(z)$ to**

I. $f(z) = 2z^3$
II. $f(z) = z^2 + 3z$
III. $f(z) = \frac{1+z}{1-z}$

**Solution**

**H.W**

**Example :**

**prove that $f(z) = |z|^2$ is differentiable function at $z = 0$**

**Solution**

$$f'(z) = \lim_{\Delta z\to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{(z+\Delta z)\overline{(z+\Delta z)} - z\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{(z+\Delta z)(\bar{z}+\overline{\Delta z}) - z\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{z\bar{z} + z\overline{\Delta z} + \bar{z}\Delta z + \Delta z\overline{\Delta z} - z\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{z\overline{\Delta z} + \bar{z}\Delta z + \Delta z\overline{\Delta z}}{\Delta z}$$

$$= \lim_{\Delta z\to 0} z\frac{\overline{\Delta z}}{\Delta z} + \bar{z} + \overline{\Delta z} \quad , \quad \because z = 0 \ \to \bar{z} = 0$$

$$= \lim_{\Delta z\to 0} (0)\frac{\overline{\Delta z}}{\Delta z} + (0) + \overline{\Delta z} \quad , \quad \because \Delta z \to 0 \quad \to \quad \overline{\Delta z} \to 0$$

$\therefore$ the derivative exist and $f'(0) = 0$

## Analytic function

### Analytic function at point

a function f(z) of a complex variable z is said to be Analytic at the point $z_0$

If it's derivative exist at this point.

### Analytic function at region

A function f(z) of a complex variable z is said to be Analytic at region if it is analytic in every point in D .

## Entire Function

A function f(z) is called entire function if it is analytic at each point in the entire plane.

## Singular Point

If f is analytic function at some points of a neighborhood of $z_0$ except at $z_0$ it self then $z_0$ is called singular point.

**Example :**

**find the singular point for $f(z) = \frac{1+z}{1-z}$**

**Solution**

$1 - z = 0 \;\rightrightarrows\; z = 1$ is singular point

$$f'(z) = \frac{2}{(1-z)^2}$$

Then f is analytic function $\forall$ z except $z = 1$

the derivative is not exists in singular point $z = 1$

**Notes**

- The polynomials are analytic function $\forall z \in \mathbb{C}$, $i.e$ it is entire function
- $if\ f(z) \& g(z)$ be two analytic functions in the region D then :
  $f(z) \pm g(z)$, $f(z) \cdot g(z)$, $f(z) o\, g(z)$, $f(z)/g(z)$, $g(z) \neq 0$

  are analytic function in D

## Cauchy Riemann Equation (C.R.E)

Suppose that $f(z) = u(x,y) + iv(x,y)$ and that $f'(z)$ exists at a point $z_0$

then the first order partial derivatives of u and v must exist at $z_0$ and must satisfy the Cauchy Riemann Equations :

$$u_x = v_y \ \& \ u_y = v_x$$

There also $f'(z)$ can be written as

$$f'(z) = u_x + iv_x \quad \text{and} \quad f'(z_0) = u_{x_0} + iv_{x_0}$$

**OR**

$$f'(z) = v_y - iu_y \quad \text{and} \quad f'(z_0) = v_{yo} - iu_{yo}$$

when these partial derivatives are to be evaluated at $z_0 = (x_0, y_0)$

**proof :**

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$f'(z) = \lim_{(\Delta x,\Delta y)\to 0} \frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta x + i\Delta y} + i\frac{v(x+\Delta x, y+\Delta y) - v(x,y)}{\Delta x + i\Delta y}$$

**the 1st path ( along the x-axis )**

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$f'(z) = u_x + i\,v_x \qquad \text{........(1)}$$

$$f'(z) = \lim_{(\Delta x,\Delta y)\to 0} \frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta x + i\Delta y} + i\frac{v(x+\Delta x, y+\Delta y) - v(x,y)}{\Delta x + i\Delta y}$$

**the 2nd path (a long the y-axis )**

$$f'(z) = \lim_{\substack{\Delta y \to 0 \\ \Delta x = 0}} \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + i\frac{v(x, y + \Delta y) - v(x, y)}{i\Delta y}$$

$$f'(z) = \lim_{\Delta y \to 0}(-i)\frac{u(x, y + \Delta y) - u(x, y)}{\Delta y} + \frac{v(x, y + \Delta y) - v(x, y)}{\Delta y}$$

$$f'(z) = -iu_y + v_y \quad \text{........(2)}$$

from (1) and (2) we get the rules of derivative , from equality of (1) and (2)

$u_x + i\,v_x = -iu_y + v_y$

$u_x = v_y$ **and** $v_x = -u_y$

**Example :**

**show that the function $f(z) = z^2$ is differentiable and satisfies The C,R,E**

**Solution**

$\mathrm{f(z) = z^2 \rightarrow f(x, y) = (x + yi)^2}$

$f(x, y) = x^2 - y^2 + i(2xy)$

$u = x^2 - y^2 \qquad v = 2xy$

$u_x = 2x \qquad v_y = 2x$

$u_y = -2y \qquad v_x = 2y$

$\therefore C.R.E$ *are satisfied*

*then* $f'(z) = u_x + iv_x$

$= 2x + i2y$

$= 2(x + iy)$

$= 2z$

**Example :**

**show that the function $f(z) = 2z^2 - 3z + 5$ is differentiable and satisfies C.R.E**

**Solution**

$f(z) = 2z^2 - 3z + 5$

$f(x, y) = 2(x + yi)^2 - 3(x + yi) + 5$

$= 2(x^2 + 2xyi - y^2) - 3x - 3yi + 5$

$= (2x^2 - 2y^2 - 3x + 5) + i(4xy - 3y)$

$u = 2x^2 - 2y^2 - 3x + 5 \qquad v = 4xy - 3y$

$u_x = 4x - 3 \qquad v_y = 4x - 3$

$u_y = -4y \qquad v_x = 4y$

$\therefore C.R.E \quad are\ satisfied$

$f'(z) = v_y - iu_y$

$= 4x - 3 - i(-4y)$

$= 4(x + iy) - 3$

$= 4z - 3$

**Example :**

**show that the function $f(z) = 2z^3$ is differentiable and satisfies C.R.E**

**Solution**

$f(x, y) = 2(x + yi)^3$

$= 2(x^3 + 3x^2yi + 3x(yi)^2 + (yi)^3)$

$= (2x^3 - 6xy^2) + i(6x^2y - 2y^3)$

$u(x,y) = 2x^3 - 6xy^2 \qquad v(x,y) = 6x^2 - 2y^3$

$u_x = 6x^2 - 6y^2 \qquad v_y = 6x^2 - 6y^2$

$u_y = -12xy \qquad v_x = 12xy$

$\therefore$ C.R.E are satisfied

$f'(z) = u_x + iv_x$

$= 6x^2 - 6y^2 + i(12xy)$

$= 6(x^2 + 2xyi - y^2)$

$= 6(x + yi)^2$

$= 6z^2$

**Example**

**By using C.R.E find $f'(z)$ *to* $f(z) = e^z$**

**Solution**

$e^z = e^{x+yi} = e^x e^{yi} = e^x[\cos y + i\sin y]$

$= e^x \cos y + i\, e^x \sin y$

$u(x,y) = e^x \cos y \qquad v(x,y) = e^x \sin y$

$u_x = e^x \cos y \qquad v_y = e^x \cos y$

$u_y = -e^x \sin y \qquad v_x = e^x \sin y$

$\therefore C.R.E$ are satisfied

$f'(z) = u_x + iv_x$

$= e^x \cos y + i\, e^x \sin y$

$= e^x[\cos y + i\sin y]$

$= e^x\, e^{yi} = e^{x+yi} = e^z$

**Example**

**By using C.R.E show that $f(z) = e^{\bar{z}}$ don't have any derivative**

**Solution**

$e^{\bar{z}} = e^{x-yi} = e^x\left[e^{-yi}\right]$

$= e^x[\cos y - i\sin y] = e^x\cos y - ie^x\sin y$

$u(x,y) = e^x\cos y \qquad v(x,y) = -e^x\sin y$

$u_x = e^x\cos y \qquad v_y = -e^x\cos y$

$\therefore$ the 1st condition of C.R.E not satisfy

then C.R.E doesn't satisfied and $f'(z)$ doesn't exist

**Example**

**is $f(z) = \cosh x\,.\cos y + i\sinh x\,.\sin y$ an entire function ?**

**Solution**

$u(x,y) = \cosh x\,.\cos y \qquad v(x,y) = \sinh x\,.\sin y$

$u_x = \sinh x\,.\cos y \qquad v_y = \sinh x\,.\cos y$

$u_y = -\cosh x\,.\sin y \qquad v_x = \cosh x\,.\sin y$

$\therefore$ C.R.E are satisfied

$f'(z) = u_x + iv_x$

$= \sinh x\,.\cos y + i\cosh x\,.\sin y$

Since $u_x, u_y, v_x$ *and* $v_y$ are exist and continuous function , and satisfy C.R.E . then $f(z)$ is entire function .

**Example :**

**is the function $f(z) = |z|^2$ is differentiable and satisfies C.R.E at $z = 0$ ?**

**Solution**

$f(x,y) = |z|^2 \quad , z_o = 0 \rightarrow (x_o, y_o) = (0,0)$

$= x^2 + y^2$

$u(x,y) = x^2 + y^2 \qquad v(x,y) = 0$

$u_x = 6x^2 - 6y^2 \rightarrow u_{xo} = 0 \qquad v_y = 6x^2 - 6y^2 \rightarrow v_{yo} = 0$

$u_y = -12xy \rightarrow u_{yo} = 0 \qquad v_x = 12xy \rightarrow v_{xo} = 0$

$\therefore$ C.R.E are satisfied

$f'(0) = u_{xo} + iv_{x0} = 0$

**Example :**

**is the following functions are differentiable and satisfies C.R.E ?**

*I.* $f(z) = z - \bar{z}$
*II.* $f(z) = iz + 2$
*III.* $f(z) = z\, Im(z)$
*IV.* $f(z) = z - i\bar{z}$
*V.* $f(z) = |z|^2 - i(|z|^2)$

**Solution**

**H.W**

**Note**

- If $f(z)$ is continuous and satisfied the C.R.E , then $f(z)$ is **analytic**.
- if $f(z)$ is : analytic , differentiable and continuous then $f(z)$ is an entire function.
- If the function is analytic at every point then these function is differentiable .

**Example :**

**prove that $f(z) = z^2 + 5iz + 3 - i$ is entire function .**

**Solution**

$f(x, y) = (x + yi)^2 + 5i(x + yi) + 3 - i$

$= x^2 + 2xyi - y^2 + 5ix - 5y + 3 - i$

$= x^2 - y^2 + -5y + 3 + i(2xy + 5x - 1)$

$u(x, y) = x^2 - y^2 - 5y + 3 \qquad v(x, y) = 2xy + 5x - 1$

$u_x = 2x \qquad v_y = 2x$

$u_y = -2y - 5 \qquad v_x = 2y + 5$

$\therefore \quad C.R.E \quad are\ satisfied$

$f'(z) = u_x + iv_x$

$= 2x + i(2y + 5)$

$= 2(x + yi) + 5i$

$= 2z + 5i$

Since the partial derivative of $u , v$ are continuous then

f(z) is entire function .

**Theorem :**

**if $f(z)$ $and$ $\overline{f(z)}$ are analytic function then $f(z)$ must be a constant**

**Proof**

let $f(z) = u + iv$ is analytic

$u_x = v_y \quad and \quad u_y = -v_x \qquad \dots\dots..(1)$

$\overline{f(z)} = u - iv$ is analytic

$u_x = -v_y \quad and \quad u_y = v_x \qquad \dots\dots..(2)$

from 1 and 2 we get

$u_x = -u_x \quad \rightarrow \quad 2u_x = 0 \quad \rightarrow \quad u_x = 0$

$u_y = -u_y \quad \rightarrow \quad 2u_y = 0 \quad \rightarrow \quad u_y = 0$

$v_x = -v_x \quad \rightarrow \quad 2v_x = 0 \quad \rightarrow \quad v_x = 0$

$v_y = -v_y \quad \rightarrow \quad 2v_y = 0 \quad \rightarrow \quad v_y = 0$

$u_x = u_y = 0 \rightarrow u = c_1$ such that $c_1$ is constant

$v_x = v_y = 0 \rightarrow v = c_2$ such that $c_2$ is constant

$\therefore f(z) = u + iv = c_1 + c_2$ is constant

**Theorem :**

**if $f(z)$ is analytic function and $|f(z)|$ is constant function , Show that $f(z)$ is a constant function .**

**Proof**

let $f(z) = u + iv$ is analytic

$u_x = v_y$ & $u_y = -v_x$ ……..(1)
since $|f(z)|$ is constant , then

$|f(z)| = |u + iv| = \sqrt{u^2 + v^2} = c$ , such that c is constant

$u^2 + v^2 = c^2$ ……..(2)

diff. (2) respect to $x$

$2u\, u_x + 2v\, v_x = 0$

$u\, u_x + v\, v_x = 0$ …….(3)

By multiply (3) by $u$

$u^2 u_x + uv\, v_x = 0$ …….(4)

diff (2) respect to y

$2u\, u_y + 2v\, v_y = 0$

$u\, u_y + v\, v_y = 0$ ……..(5)

By multiply (5) by $v$

$uv\, u_y + v^2\, v_y = 0$ ……..(6)

adding the equations (4) + (6)
$u^2 u_x + uv\, v_x + uv\, u_y + v^2\, v_y = 0$ ……….(7)

since function is analytic , then it is satisfy C.R.E ($u_x = v_y$ & $u_y = -v_x$)

$u^2 u_x + uv(-u_y) + uv\, u_y + v^2 u_x = 0$

$u_x(u^2 + v^2) = 0$

$c^2 . u_x = 0 \rightarrow \boxed{\boldsymbol{u_x = 0}}$

$u^2 u_x + uv\, v_x + uv\, u_y + v^2\, v_y = 0$ ……….(7)

since function is analytic , then it is satisfy C.R.E ($u_x = v_y$ & $u_y = -v_x$)

$u^2 v_y + uv\, v_x + uv(-v_x) + v^2 v_y = 0$

$v_y(u^2 + v^2) = 0$

$c^2 . v_y = 0 \;\rightarrow\; \boxed{\boldsymbol{v_y = 0}}$

By multiply (3) by $v$

$uv\, u_x + v^2\, v_x = 0$ …….(8)

By multiply (5) by $u$

$u^2\, u_y + uv\, v_y = 0$ ……..(9)

subtracting : (8)-(9)

$\mathrm{uvu_x + v^2v_x - u^2u_y - uvv_y = 0}$ ……..(10)

since function is analytic , then it is satisfy C.R.E ($u_x = v_y$ & $u_y = -v_x$)

$\mathrm{uvu_x + v^2v_x - u^2}(-v_x) - \mathrm{uv}(u_x) = 0$

$(u^2 + v^2)\mathrm{v_x} = 0$ , $c^2 . \mathrm{v_x} = 0 \;\rightarrow\; \boxed{\boldsymbol{v_x = 0}}$

$\mathrm{uvu_x + v^2v_x - u^2u_y - uvv_y = 0}$ ……..(10)

since function is analytic , then it is satisfy C.R.E ($u_x = v_y$ & $u_y = -v_x$)

$\mathrm{uvu_x - v^2u_y - u^2(u_y) - uv}(u_x) = 0$

$-(u^2 + v^2)\mathrm{u_y} = 0$ , $-c^2 . \mathrm{u_y} = 0 \;\rightarrow\; \boxed{\boldsymbol{u_y = 0}}$

$\because\; u_x, u_y, v_x, v_y = 0 \rightarrow u, v\ are\ constant$

$\therefore f(z) = u + iv = c_1 + ic_2$ is constant