Harmonic function

a real valued function h(x, y) of two real variable x *and* y is said to be harmonic in a given domain of xy- plane If through that domain it has a continuous first & Second partial derivatives and satisfies Laplace equation

$$\nabla^2 h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$$

Theorem :

if $f(z) = u(x, y) + i\, v(x, y)$ is analytic in a region D , then two functions u & v are harmonic ?

Proof

since f(z) is analytic then C.R.E are satisfied for all z in domain D .

$u_x = v_y$ ....(1)

$u_y = -v_x$ ....(2)

By differentiating both sides in (1)&(2) respect to x we get

$u_{xx} = v_{yx}$
$u_{yx} = -v_{xx}$ .... (3)

Also differentiating both sides in (1)&(2) respect to y we get

$u_{xy} = v_{yy}$
$u_{yy} = -v_{xy}$ .... (4)

Since $u_{xy} = u_{yx}$ & $v_{xy} = v_{yx}$

we get from (3)&(4)

$v_{yy} = -v_{xx}$ & $u_{yy} = -u_{xx}$

$v_{yy} + v_{xx} = 0$ & $u_{yy} + u_{xx} = 0$

$\therefore\ \nabla^2 u = 0$ & $\nabla^2 v = 0$

$\therefore u$ & v are harmonic

Note

the converse of theorem is not true

Example

prove that $f(z) = 2xy + i(x^2 - y^2)$ is harmonic But not analytic.

Solution

$u(x, y) = 2xy \qquad v(x, y) = x^2 - y^2$

$u_x = 2y \qquad v_y = -2y$

$u_y = 2x \qquad v_x = 2x$

C.R.E don't satisfied

Then the function not analytic

$u_{xx} = 0 \qquad u_{yy} = 0$

$v_{xx} = 2 \qquad v_{yy} = -2$

$\nabla^2 u = u_{xx} + u_{yy} = 0 + 0 = 0$

$\nabla^2 v = v_{xx} + v_{yy} = 2 - 2 = 0$

The function $f(z)$ is harmonic.

Example

is the function $u = sinx\ cosy$ is harmonic ?

Solution

$u_x = cosx\ cosy$

$u_{xx} = -sinx\ cosy$

$u_y = -sinx\ siny$

$u_{yy} = -sinx\ cosy$

$u_{xx} + u_{yy} = -sinx\ cosy - sinx\ cosy \neq 0$

$\therefore u$ is not harmonic

Example

is the function $u = e^{-x}(x\ siny - y\ cosy)$ is harmonic ?

Solution

$u = xe^{-x}\ siny - e^{-x}y\ cosy$

$u_x = siny[-xe^{-x} + e^{-x}] + e^{-x}y\ cosy$

$u_{xx} = siny[xe^{-x} - e^{-x} - e^{-x}] - e^{-x}y\ cosy$

$u_{xx} = xe^{-x}siny - 2e^{-x}siny - e^{-x}y\ cosy$ **……..(1)**

$u_y = xe^{-x}\ cosy - e^{-x}(-ysiny + cosy)$

$u_{yy} = -xe^{-x}\ siny - e^{-x}(-ycosy - siny - siny)$

$u_{yy} = -xe^{-x}\ siny + e^{-x}ycosy + 2e^{-x}siny$ **……..(2)**

$\nabla^2 u = u_{xx} + u_{yy}$

$= xe^{-x}siny - 2e^{-x}siny - e^{-x}y\ cosy - xe^{-x}\ siny + e^{-x}ycosy + 2e^{-x}siny$

$\nabla^2 u = 0$

$\therefore u$ is harmonic

Example

is the function $f(z) = ln|z|^2$ is harmonic ?

Solution

$$f(z) = ln|z|^2 = \ln(x^2 + y^2)$$

$$f_x = \frac{2x}{x^2 + y^2}$$

$$f_{xx} = \frac{2(x^2 + y^2) - 2x \cdot 2x}{(x^2 + y^2)^2} = \frac{2y^2 - 2x^2}{(x^2 + y^2)^2}$$

$$f_y = \frac{2y}{x^2 + y^2}$$

$$f_{yy} = \frac{2(x^2 + y^2) - 2y \cdot 2y}{(x^2 + y^2)^2} = \frac{2x^2 - 2y^2}{(x^2 + y^2)^2}$$

$$\nabla^2 f = f_{xx} + f_{yy} = \frac{2y^2 - 2x^2}{(x^2 + y^2)^2} + \frac{2x^2 - 2y^2}{(x^2 + y^2)^2} = 0$$

$\therefore f$ is harmonic

Example

is the function $T(x, y) = e^{-y} \sin x$ is harmonic ?

Solution

$$T_x = e^{-y} \cos x \quad \rightarrow \quad T_{xx} = -e^{-y} \sin x$$

$$T_y = -e^{-y} \sin x \quad \rightarrow \quad T_{yy} = e^{-y} \sin x$$

$$\nabla^2 T = T_{xx} + T_{yy} = -e^{-y} \sin x + e^{-y} \sin x = 0$$

$\therefore$ T is harmonic.

Example

The function $f(z) = e^{-y} \sin x - i\, e^{-y} \cos x$ is entire , show that the component function u and v are harmonic in its domain D .

Solution

H.W

Example

The function $f(z) = \frac{i}{z^2}$ is analytic whenever z $\neq$ 0 ,then the two functions u & v are harmonic throughout any domain in the xy − plane that doesn't contain the origin .

Solution

H.W

Harmonic Conjugate

- if two given functions u and v are harmonic in a domain D and there is order partial derivatives satisfy the C.R.E throughout D, then v is said to be harmonic conjugate for u
- a function $f(z) = u(x,y) + i\, v\,(x,y)$ is analytic in a domain D iff v is harmonic conjugate of u .
- if v is harmonic conjugate of u in some domain, it is not in general true that u is a harmonic conjugate of v.
- when give u is harmonic function and ask. To find the analytic function F, this mean that we will find the harmonic conjugate v

Example

Find the harmonic conjugate function v for a harmonic function $u = y^3 - 3x^2y$

Solution

since a harmonic conjugate function $v(x,y)$ is related to $u(x,y)$, then by C.R.E

$u_x = v_y \quad , \quad u_y = -v_x$

$u_x = -6xy$

$v_y = -6xy$, by integral respect to y

$v = -3xy^2 + \emptyset(x)$ ....(1)

diff. (1) respect to x

$v_x = -3y^2 + \emptyset'(x)$ ....(2)

but $v_x = -u_y = -(3y^2 - 3x^2)$

$v_x = 3x^2 - 3y^2$ ....(3)

Sub. (3) in (2)

$3x^2 - 3y^2 = -3y^2 + \emptyset'(x) \quad \rightarrow \quad \emptyset'(x) = 3x^2$

$$\emptyset(x) = \int \emptyset'(x)\,dx = \int 3x^2\,dx = x^3 + c$$

From (1) we get

$$v(x,y) = -3xy^2 + x^3 + c$$

and the analytic function is

$$f(z) = (y^3 - 3x^2y) + i(-3xy^2 + x^3 + c)$$

Example

Find the harmonic conjugate function v for $u = 2x(1-y)$

Solution

$u_x = 2(1-y) \quad \rightarrow \quad u_{xx} = 0$

$u_y = -2x \quad \rightarrow \quad u_{yy} = 0$

$u_{xx} + u_{yy} = 0$

$\therefore u$ is not harmonic.

$u_x = v_y \;\rightarrow\; v_y = 2(1-y) \quad by \int dy$

$v = 2y - y^2 + \emptyset(x)$ ....(1)

diff.(1) respect to x

$v_x = \emptyset'(x)$

But $v_x = -u_y = 2x$

$2x = \emptyset'(x) \;\rightarrow\; \emptyset(x) = x^2 + c$

$v(x,y) = 2y - y^2 + x^2 + c$

Example

Find the harmonic conjugate function v , if

I. $u = 2x - x^3 + 3xy^2$

II. $u = \sinh x \sin y$

III. $u = \frac{y}{x^2+y^2}$

Solution

H.W

Example

prove that $u(x,y) = e^x\, cosy$ is harmonic function and find harmonic conjugate

Solution

$u_x = e^x\, cosy \quad \rightarrow \quad u_{xx} = e^x\, cosy$

$u_y = -e^x\, siny \quad \rightarrow \quad u_{yy} = -e^x\, cosy$

$u_{xx} + u_{yy} = e^x\, cosy - e^x\, cosy = 0$

$\therefore u$ is harmonic

$u_x = v_y \quad \rightarrow \quad v_y = e^x\, cosy \qquad by \int dy$

$v = e^x\, siny + \emptyset(x) \quad ....(1)$

diff.(1) respect to x

$v_x = e^x\, siny + \emptyset'(x)$

but $\quad v_x = -u_y = e^x\, siny$

$e^x\, siny = e^x\, siny + \emptyset'(x) \rightarrow \emptyset'(x) = 0 \rightarrow \quad \emptyset(x) = C$

$v(x,y) = e^x\, siny + C$

Theorem :

if v_1 & v_2 are two harmonic conjugate to a function u in the region D , then they are different by a constant

Proof

Since v_1 is harmonic conjugate to u

$f(z) = u + iv_1$ is analytic

$u_x = v_1y \quad \& \quad u_y = -v_1x \quad ....(1)$

Also Since v_2 is harmonic conjugate to u

$f(z) = u + iv_2$ is analytic

$u_x = v_2y$ & $u_y = -v_2x$ ....(2)

from (1)& (2) we get

$v_1y = v_2y$ by $\int dy$

$v_1 = v_2 + C$

$v_1 - v_2 = C$

Simple Curves & Orthogonal Curves

if the function f(z) contain two components u & v then we said to the set of functions u & v that is Simple curves $f(x, y) = c$, and this curves be orthogonal if the multiply of the slope of them is (-1)

Note : the slope is the 1st derivative : $m = \frac{dy}{dx}$

Example

Find the simple curves and prove it is orthogonal : $f(z) = z^2 + 5zi + 3 - i$

Solution

$f(x, y) = (x^2 - y^2 - 5y + 3) + i\,(2xy + 5x - 1)$

$u = x^2 - y^2 - 5y + 3 \rightarrow x^2 - y^2 - 5y = -3$ is simple curve

$v = 2xy + 5x - 1 \rightarrow 2xy + 5x = 1$ is simple curve

the functions u & v are simple curves to prove the orthogonality we will find the slope to the fun curves

$m_1 : 2x - 2y\frac{dy}{dx} - 5\frac{dy}{dx} + 0 = 0$

$\frac{dy}{dx}(-2y-5) = -2x$

$\frac{dy}{dx} = \frac{2x}{2y+5} = m_1$ *(1st slope)*

$m_2 : 2x\frac{dy}{dx} + 2y + 5 - 0 = 0$

$\frac{dy}{dx} = \frac{-2y-5}{2x} = m_2$ 2nd slope

$m_1\, m_2 = \frac{2x}{2y+5}\frac{-2y-5}{2x} = -1$

$\therefore u$ & v orthogonal curves

Polar Coordinates

$z = x + yi \quad or \quad z = re^{i\theta} \qquad (z \neq 0)$

$z = r\cos\theta + ir\sin\theta$

$x = r\cos\theta \qquad , \qquad y = r\sin\theta$ ……………(1)

$f(z) = u(x,y) + iv(x,y)$

we will transform by r, θ *i.e*

$f(z) = u(r,\theta) + iv(r,\theta)$

then the 1st derivatives of u & v with respect to r *and* θ ,and by chain rule

$\frac{du}{dr} = \frac{du}{dx}\frac{dx}{dr} + \frac{du}{dy}\frac{dy}{dr}$

$\frac{du}{d\theta} = \frac{du}{dx}\frac{dx}{d\theta} + \frac{du}{dy}\frac{dy}{d\theta}$

Can write as

$u_r = u_x\cos\theta + u_y\sin\theta$

$u_\theta = -u_x r\sin\theta + u_y r\cos\theta$

$$\left.\begin{aligned} u_r &= u_x \cos\theta + u_y \sin\theta \\ u_\theta &= r[-u_x \sin\theta + u_y \cos\theta] \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (2)$$

Like wise

$$v_r = v_x \cos\theta + v_y \sin\theta$$

$$v_\theta = -v_x r \sin\theta + v_y r \cos\theta$$

$$\left.\begin{aligned} v_r &= v_x \cos\theta + v_y \sin\theta \\ v_\theta &= r[-v_x \sin\theta + v_y \cos\theta] \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (3)$$

if partial derivatives of u and v with respect to x and y also satisfy C.R.E

$$u_x = v_y \quad \& \quad u_y = -v_x \ldots\ldots\ldots\ldots (4)$$

eq. (3) becomes

$$\left.\begin{aligned} v_r &= -u_y \cos\theta + u_x \sin\theta \\ v_\theta &= r[u_y \sin\theta + u_x \cos\theta] \end{aligned}\right\} \ldots\ldots\ldots\ldots (5)$$

From (2) & (5) we get

$$\boxed{v_\theta = r u_r} \quad \& \quad \boxed{u_\theta = -r v_r} \ldots\ldots\ldots\ldots (6)$$

Eq(6) called C.R.E in polar coordinates

Theorem :

let the function $f(z) = u(r,\theta) + iv(r,\theta)$ be defined throughout some ε neighbourhood anon point to z_0 , and suppose that :

1- the 1st order partial derivatives of the functions u *and* v with respect to r *and* θ exists every where in the neighbourhood

2- these partial derivatives are continuous at $(r_0 , \ \theta_0)$ and satisfy the polar form; $v_\theta = r u_r$ & $u_\theta = -r v_r$, Then $f'(z)$ exists and its value is

$$f'(z) = e^{-i\theta}(u_r + i v_r) \ldots\ldots\ldots\ldots (1)$$

$$\boldsymbol{f'(z) = \frac{ie^{-i\theta}}{r}(u_\theta + iv_\theta) \ldots\ldots\ldots\ldots (2)}$$

Proof

$f(z) = u(x,y) + iv(x,y) \quad , \ z = re^{i\theta}$

$f(z) = u(r,\theta) + iv(r,\theta)$

Diff. respect to r

$f'(z)\frac{dz}{dr} = u_r + iv_r$

$f'(z)e^{i\theta} = u_r + iv_r$

$$\therefore \boldsymbol{f'(z) = e^{-i\theta}(u_r + iv_r)}$$

$f(z) = u(x,y) + iv(x,y) \quad , \ z = re^{i\theta}$

$f(z) = u(r,\theta) + iv(r,\theta)$

Diff. respect to θ

$f'(z)\frac{dz}{d\theta} = u_\theta + iv_\theta$

$f'(z)\ ire^{i\theta} = u_\theta + iv_\theta$

$$\boldsymbol{f'(z) = \frac{e^{-i\theta}}{ir}(u_\theta + iv_\theta) = \frac{-ie^{-i\theta}}{r}(u_\theta + iv_\theta)}$$

Example

Find the derivative of $f(z) = \frac{1}{z}$ in polar form

Solution

$f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$

$$=\frac{1}{r}[\cos\theta - i\sin\theta]$$

$$=\frac{1}{r}\cos\theta - i\frac{1}{r}\sin\theta$$

$u=\frac{1}{r}\cos\theta$, $v=-\frac{1}{r}\sin\theta$

$u_r=\frac{-1}{r^2}\cos\theta$, $v_r=\frac{1}{r^2}\sin\theta$

$u_\theta=\frac{-1}{r}\sin\theta$, $v_\theta=\frac{-1}{r}\cos\theta$

$$\therefore f'(z)=e^{-i\theta}(u_r+iv_r)$$

$$=e^{-i\theta}\left(\frac{-1}{r^2}\cos\theta+i\frac{1}{r^2}\sin\theta\right)$$

$$=\frac{-e^{-i\theta}}{r^2}(\cos\theta-i\sin\theta)$$

$$=\frac{-e^{-2i\theta}}{r^2}=\frac{-1}{r^2e^{2i\theta}}=\frac{-1}{(re^{i\theta})^2}=-\frac{1}{z^2}$$

Another method

$$f'(z)=\frac{-ie^{-i\theta}}{r}(u_\theta+iv_\theta)$$

$$=\frac{-ie^{-i\theta}}{r}\left[\frac{-1}{r}\sin\theta-\frac{i}{r}\cos\theta\right]$$

$$=\frac{e^{-i\theta}}{r^2}(i\sin\theta-\cos\theta)=-\frac{e^{-2i\theta}}{r^2}=-\frac{1}{r^2e^{2i\theta}}=\frac{-1}{(re^{i\theta})^2}=-\frac{1}{z^2}$$

Example

Consider the function $g(z)=\sqrt{r}e^{i\frac{\theta}{2}}$, show that $g(z)$ has a derivative at each point in its domain and $g'(z)=\frac{1}{2g(z)}$

Solution

$$g(z)=r^{\frac{1}{2}}(cos\frac{\theta}{2}+i\,sin\frac{\theta}{2})$$

$$=r^{\frac{1}{2}}cos\frac{\theta}{2}+ir^{\frac{1}{2}}sin\frac{\theta}{2}$$

$$u=r^{\frac{1}{2}}cos\frac{\theta}{2} \quad , \quad v=r^{\frac{1}{2}}sin\frac{\theta}{2}$$

$$u_r=\frac{1}{2}r^{-\frac{1}{2}}cos\frac{\theta}{2} \quad , \quad v_\theta=\frac{1}{2}r^{\frac{1}{2}}cos\frac{\theta}{2}$$

$$u_\theta=-\frac{1}{2}r^{\frac{1}{2}}sin\,\theta \quad , \quad v_r=\frac{1}{2}r^{-\frac{1}{2}}sin\frac{\theta}{2}$$

$\therefore g'(z)$ exists

$$g'(z)=e^{-i\theta}(u_r+iv_r)$$

$$=e^{-i\theta}(\frac{1}{2}r^{-\frac{1}{2}}cos\frac{\theta}{2}+i\frac{1}{2}r^{-\frac{1}{2}}sin\frac{\theta}{2})$$

$$=e^{-i\theta}\frac{1}{2\sqrt{r}}(cos\frac{\theta}{2}+i\,sin\frac{\theta}{2})$$

$$=e^{-i\theta}\frac{1}{2\sqrt{r}}(cos\frac{\theta}{2}+i\,sin\frac{\theta}{2})$$

$$=e^{-i\theta}\frac{1}{2\sqrt{r}}e^{\frac{i\theta}{2}}$$

$$g'(z)=\frac{1}{2\sqrt{r}e^{i\frac{\theta}{2}}}=\frac{1}{2g(z)}$$

Example

Verify C.R.E and $f'(z)$ by polar

1- $f(z) = \sqrt[3]{r}e^{i\frac{\theta}{3}}$

2- $f(z) = \frac{1}{z^4} \quad ,(z \neq 0)$

Solution

H.W

Example

write Laplace equation in polar form

Solution

from C.R.E

$v_\theta = r u_r \ldots\ldots\ldots\ldots..(1)$

$u_\theta = -r v_r \;\ldots\ldots\ldots..(2)$

Diff (1) respect to r & (2) respect to θ

$r\, u_{rr} + u_r = v_{\theta r} \ldots\ldots\ldots..(3)$

$-r\, v_{r\theta} = u_{\theta\theta} \ldots\ldots\ldots\ldots..(4)$

$\because\; v_{\theta r} = v_{r\theta} = \frac{-1}{r} u_{\theta\theta}$, sub. in (3)

$r u_{rr} + u_r = \frac{-1}{r} u_{\theta\theta}$ mult. By r

$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0 \ldots\ldots\ldots..(A)$

To find v :

Diff (1) respect to θ & (2) respect to r

$r u_{r\theta} = v_{\theta\theta} \ldots\ldots\ldots\ldots (5)$

$-r v_{rr} - v_r = u_{\theta r} \ldots\ldots\ldots\ldots..(6)$

$\because u_{\theta r} = u_{r\theta} = \frac{1}{r} v_{\theta\theta}$

$-r v_{rr} - v_r = \frac{1}{r} v_{\theta\theta}$ mult. By $-r$

$r^2 v_{rr} + r v_r + v_{\theta\theta} = 0 \ldots\ldots\ldots..(B)$

Eq.(A) & (B) called Laplace eq. in polar form.

$$\boldsymbol{r^2 u_{rr} + r u_r + u_{\theta\theta} = 0}$$
$$\boldsymbol{r^2 v_{rr} + r v_r + v_{\theta\theta} = 0}$$

Example

Show that the function $f(z) = z^2 - 3 + i$ satisfy Laplace equation in polar form

Solution

$f(r,\theta) = r^2\cos(2\theta) + i\, r^2 sin(2\theta) - 3 + i$

$u(r,\theta) = r^2\cos(2\theta) - 3$, $v(r,\theta) = r^2 sin(2\theta) + 1$

$u_r = 2rcos(2\theta)$, $u_{rr} = 2rcos(2\theta)$

$u_\theta = -2r^2\cos(2\theta)$, $u_{\theta\theta} = -2r^2\sin(2\theta)$

$v_r = 2rsin(2\theta)$, $u_{rr} = 2sin(2\theta)$

$v_\theta = 2r^2\cos(2\theta)$, $v_{\theta\theta} = -4r^2\sin(2\theta)$

$\boldsymbol{r^2 u_{rr} + r u_r + u_{\theta\theta} = 0}$

$2r^2\cos(2\theta) + 2r^2\cos(2\theta) - 4r^2\cos(2\theta) = 0$

$\boldsymbol{r^2 v_{rr} + r v_r + v_{\theta\theta} = 0}$

$2r^2\sin(2\theta) + 2r^2\sin(2\theta) - 4r^2\sin(2\theta) = 0$

then satisfy Laplace equation in polar coordinates

Example

Show that $f(z) = 2z^2 - 5z + 2 - 4i$ satisfy Laplace equation in polar form

Solution

H.W