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complex analysis

2nd Course

Chapter 3

Dr.Mohammed Sabah Altaee

Chapter Three

Residues & Poles

Singular points

the point z_0 is called a singular point of a function f if f is falls to be analytic at z_0 but is analytic at some point in every neighborhood of z_0 , singular point z_0 can be classified into three types:

- **isolated point**

  is the point that can be isolated from principle function i.e there is a deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0 throughout which f is analytic .

- **Essential points**

  is point that can't isolated from the principle function so as the example above

- **Removable point**

  Is the point can be deleted or removable by factoring or simplify

Example

find the singular points and determine its type

$\mathbf{f(z) = \frac{1}{z-1}}$

Solution

$z - 1 = 0 \rightarrow z = 1$ is singular point the point is **isolated** because every deleted neighborhood of it contain point.

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$\mathbf{f(z) = \frac{1}{z^3(z^2+1)}}$

Solution

the singular points $z = 0$, $\mp i$ are isolated point because can draw a neighborhood contain the points.

$\mathbf{f(z) = \log z}$

Solution

the Singular point is zero and all negative values, and this points can't draw a neighborhood contains the values , and its essential.

$$\boldsymbol{f(z) = \frac{z^2 - 1}{z - 1}}$$

the point $z_0 = 1$ is Removable because

$f(z) = \frac{(z+1)(z-1)}{(z-1)} = z + 1$

$$\boldsymbol{f(z) = \frac{z^2 - 1}{z^2 - 3z + 2}}$$

Solution

H.W

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Singular Points Test (Laurent)

expand the function by Laurent series as follows

$$f(z) = \underbrace{-\infty + \cdots + \frac{b_2}{(z-z_0)^2} + \frac{b_1}{(z-z_0)}}_{P.P} + \underbrace{a_0 + a_1(z-z_0) + \cdots + \infty}_{A.P}$$

- the positive part from a function f is called the Analytic part A.P
- the negative part from a function f is called the principle part P.P
- if $\mathbf{P.P = 0}$ (no negative terms) then the point called (**Removable**)
- if $\mathbf{P.P}$ is exists in all terms then the point called (**Essential**).
- if $\mathbf{P.P}$ is exist in a determinate number of terms , then the point is called (**isolated**) , and the power of Point is the order the singular point.

Example

Find the Singular point and determine its type

$$f(z) = \frac{1-\cos z}{z}$$

Solution

$$\frac{1-\cos z}{z} = \frac{1}{z}\left[1-\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots\right)\right]$$

$$= \frac{1}{z}\left(1-1+\frac{z^2}{2!}-\frac{z^4}{4!}+\frac{z^6}{6!}-\cdots\right)$$

$$= \frac{z}{2!}-\frac{z^3}{4!}+\frac{z^5}{6!}+\cdots$$

The $P.P = 0$

Then the singular point $z_0 = 0$ is removable

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$$f(z)=\frac{e^z-1}{z^2}$$

Solution

$$=\frac{1}{z^2}\left(1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots-1\right)$$

$$=\frac{z}{1}+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots$$

$P.P$ is exist at one term then the point is isolated and the pole is simple

$$f(z)=ze^{1/z}$$

Solution

$$=z\left[1+\frac{1}{z\,1!}+\frac{1}{z^2\,2!}+\frac{1}{z^3\,3!}+\cdots\right]$$

$$=\left[z+1+\frac{1}{2!\,z}+\frac{1}{3!\,z^2}+\cdots\right]$$

P. P is exist in all terms then the singular point $z_0=0$ is essential .

$$f(z)=\frac{\cos(z+i)-1}{(z+i)^4}$$

Solution

$z_0=-i$ is called removable

$$\frac{1}{(z+i)^4}\left[1-\frac{(z+i)^2}{2!}+\frac{(z+i)^4}{4!}-\cdots-1\right]$$

$$=\frac{1}{2!\,(z+i)^2}+\frac{1}{4!}-\frac{(z+i)^2}{6!}+\cdots$$

The singular point $z_0=-i$ is isolated

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$$f(z) = \cos(z^2 + z^{-2})$$

Solution

$$1 - \frac{1}{2!}(z^2 + z^{-2})^2 + \frac{1}{4!}(z^2 + z^{-2})^4 + \cdots$$

$$1 - \frac{1}{2!}(z^4 + 2 + z^{-4}) + \frac{1}{4!}(\ldots)$$

$$= 1 - \frac{z^4}{2!} - 1 - \frac{1}{2!\,z^4} + \cdots$$

P. P is many terms only (not all) i. the singular point is isloated

$$f(z) = sin(\frac{1}{z})$$

Solution **H.W**

$$f(z) = sin(z) + cos(z)$$

Solution **H.W**

$$f(z) = e^z + e^{-z}$$

Solution **H.W**

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The Residues

Definition

Let z_0 an isolated singular point of function f, the complex number b_1 which is the coefficient of $(z-z_0)^{-1}$ in laurent series expansion

$$f(z) = -\infty + \cdots + \frac{b_2}{(z-z_0)^2} + \frac{\boxed{b_1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots + \infty$$

b_1 is called the **Residues** of f at the isolated singular point z_0 ,

we shall write if :

$$\boldsymbol{Res[f, z_0] = b_1}$$

Theorem

let C be a simple closed counter, described in the $+ive$ Sense , if a function f is analytic inside and on c except for a finite number of singular points

$z_k = (k = 1,2, \ldots, n)$ inside c then

$$\boldsymbol{\int_c f(z)dz = 2\pi i \left[\sum_{i=i}^{n} Res[f, z_i]\right]}$$

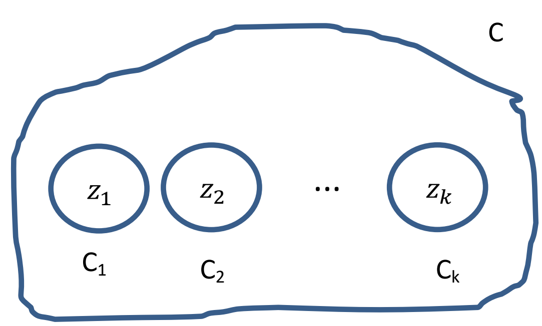
Proof:-

let the points $z_k : (k = 1,2, \ldots, n)$

be a center of positively oriented circle c_k

which are interior on c and are so one of

them have points in common



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By cauchy-corsat theorem ,To such domains

$$\int_c f(z)dz = \int_{c_1} f(z)dz + \int_{c_2} f(z)dz + \cdots + \int_{c_k} f(z)dz$$

by Cauchy corset theorem

$$\int_c f(z)dz = 2\pi i\, f(z_1) + 2\pi i f(z_2) + \cdots + 2\pi i\, f(z_k)$$

but $f(z_i) = Res[f, z_i]$

$$\int_c f(z)dz = 2\pi i\, Res[f, z_1] + 2\pi i\, Res[f, z_2] + \cdots + 2\pi i\, Res[f, z_k]$$

$$\int_c f(z)dz = 2\pi i\left[\sum_{i=1}^{k} Res[f, z_i]\right]$$

Example

Find the residues of function $f(z) = \frac{e^{-z}}{(z-1)^2}$ and find $\int_c f(z)dz$

Solution

The singular point $z_0 = 1$ then

$$\frac{e^{-z}}{(z-1)^2} = \frac{e^{-(z-1+1)}}{(z-1)^2} = \frac{e^{-(z-1)-1}}{(z-1)^2}$$

$$= \frac{e^{-1}}{(z-1)^2}\, e^{-(z-1)}$$

$$= \frac{e^{-1}}{(z-1)^2}\left(1 - (z-1) + \frac{(z-1)^2}{2!} - \frac{(z-1)^3}{3!} + \cdots\right)$$

$$= \frac{e^{-1}}{(z-1)^2} - \frac{e^{-1}}{(z-1)} + \frac{e^{-1}}{2!} - \frac{e^{-1}\,(z-1)}{3!} + \cdots$$

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$$Res[f,1] = b_1 = -e^{-1}$$

$$\oint_c f(z)dz = 2\pi i(-e^{-1}) = -2\pi i\, e^{-1}$$

Example

Find the value integral $\int_c e^{1/z^2}$ c: $|z| = 1$

Solution

$$e^{1/z^2} = 1 + \frac{(1/z^2)}{1!} + \frac{(1/z^2)^2}{2!} + \cdots$$

$$= 1 + \frac{1}{z^2} + \frac{1}{2z^4} + \cdots$$

$$b_1 = 0$$

$$\oint_c f(z)dz = 2\pi i(0) = 0$$

Example

Find the value integral $\oint_c \frac{\sin z}{z^2} dz$

Solution

$$\frac{1}{z^2}\sin z = \frac{1}{z^2}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)$$

$$= \frac{1}{z} - \frac{z}{6} + \frac{z^3}{5!} - \cdots$$

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