$b_1 = 1$

$\oint_c \frac{\sin z}{z^2} dz = 2\pi i(1) = 2\pi i$

Example

find $\int_c \frac{5z-2}{z(z-1)} dz \quad , \quad |z| > 1$

Solution

$$\frac{5z-2}{z} \frac{1}{z-1} = \frac{5z-2}{z} \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$\frac{5z-2}{z^2}\left(\frac{1}{1-\frac{1}{z}}\right) = \left(\frac{5}{z} - \frac{2}{z^2}\right)\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right)$$

$$= \frac{5}{z} + \frac{5}{z^2} + \frac{5}{z^3} + \cdots - \frac{2}{z^2} - \frac{2}{z^3}$$

$b_1 = 5$

$\oint_c f(z)dz = 2\pi i(5) = 10\pi i$

The poles

Definition

the function $f(z)$have a pole from the order (m) on z_0 if :-

$$\lim_{z\to z_0} \phi(z_0) \neq 0$$

Such that :

$$\phi(z) = (z - z_0)^m f(z)$$

We can calculate the Residues by short secant Rule

$$Res[f, z_0] = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}} (\phi(z))$$

Example

Find pole order for the function $f(z) = \frac{e^{-2z}}{z^3}$, find the Residues

Solution

$$\phi(z) = (z - z_0)^m f(z)$$

$$z_0 = 0 \quad , \quad m = 3$$

$$\lim_{z\to 0}(z - 0)^3 \frac{e^{-2z}}{z^3} = \lim_{z\to 0} e^{-2z} = 1 \neq 0$$

$\therefore$ the order is 3

$$Res[f, 0] = \frac{1}{(3-1)!} \lim_{z\to 0} \frac{d^2}{dz^2} e^{-2z}$$

$= \lim_{z\to 0} \frac{1}{2}\,(4e^{-2z}) = 2$

Example

Find the order of the pole of the $f(z) = \frac{1-e^z}{z^3}$, find the Residues

Solution

$z_0 = 0 \quad , \quad m = 3$

$\lim_{z\to 0}(z-0)^3 \frac{1-e^z}{z^3} = \lim_{z\to 0}(1 - e^{-2z}) = 0$

$\therefore m = 2 \qquad [\because m = 3 \;\; don't\; satisfied]$

$\lim_{z\to 0}(z-0)^2 \;\frac{1-e^z}{z^3} = \lim_{z\to 0} \frac{1-e^z}{z}$

$\lim_{z\to 0} \frac{-e^z}{1} = -1 \neq 0$

$Res[f,0] = \frac{1}{(2-1)!}\;\; \lim_{z\to 0} \frac{d}{dz}\, z^2\, \frac{1-e^z}{z^3}$

$= \lim_{z\to 0} \frac{d}{dz}\, \frac{1-e^z}{z}$

$= \lim_{z\to 0} \frac{d}{dz}\, \frac{1}{z}\left(1 - 1 - z - \frac{z^2}{2!} - \frac{z^3}{3!} - \cdots\right)$

$= \lim_{z\to 0} \frac{d}{dz}\, \frac{1}{z}\left(-z - \frac{z^2}{2!} - \frac{z^3}{3!} - \cdots\right)$

$= \lim_{z\to 0} \frac{d}{dz}\left(-1 - \frac{z}{2!} - \frac{z^2}{3!} - \cdots\right)$

$= \lim_{z\to 0}\;\left(-\frac{1}{2} - \frac{z}{3} - \cdots\right) = -\frac{1}{2}$

Example

Determine the order of the pole fun. $f(z) = \frac{z+1}{z^2+9}$

Solution

$$f(z) = \frac{z+1}{z^2+9} = \frac{z+1}{(z-3i)(z+3i)}$$

Case1 :- $z_0 = 3i \quad , \quad m = 1$

$$\lim_{z\to 3i}(z-3i)\,\frac{z+1}{(z-3i)(z+3i)}$$

$$\lim_{z\to 3i}\frac{z+1}{z+3i} = \frac{1+3i}{6i} = \boxed{\frac{3-i}{6}}$$

Case2 :- $z_0 = -3i \quad , \quad m = 1$

$$\lim_{z\to -3i}(z+3i)\,\frac{z+1}{(z-3i)(z+3i)}$$

$$\lim_{z\to -3i}\frac{z+1}{z-3i} = \frac{-3i+1}{-6i} = \boxed{\frac{3+i}{6}}$$

Note:-

If $m = 1$ then

$$\boxed{Res[f, z_0] = \lim_{z\to z_0}\phi(z)}$$

Example

find the Residues using poles $f(z) = \frac{z^2+1}{(z-1)(z-2)^2(z-3)}$

Solution

the singular points $\begin{pmatrix} z_0 = 1 & m = 1 \\ z_0 = 2 & m = 1 \\ z_0 = 3 & m = 1 \end{pmatrix}$

Case1 :- $z_0 = 1 \quad , \quad m = 1$

$\lim_{z\to 1}(z-1)\frac{(z^2+1)}{(z-1)(z-2)^2(z-3)}$

$\lim_{z\to 1}\frac{(z^2+1)}{(z-2)^2(z-3)} = -1$

$\therefore Res[f,1] = -1 \qquad [\because m = 1]$

Case2 :- $z_0 = 2 \quad , \quad m = 2$

$\lim_{z\to 2}(z-2)^2\frac{(z^2+1)}{(z-1)(z-2)^2(z-3)} = -5 \neq 0$

$Res[f,2] = \frac{1}{1!}\lim_{z\to 2}\frac{d}{dz}\frac{(z^2+1)}{(z-1)(z-3)}$

$= \lim_{z\to 2}\frac{d}{dz}\frac{(z^2+1)}{z^2-4z+3}$

$= \lim_{z\to 2}\frac{d}{dz}\frac{(z^2-4z+3)(2z)-(z^2+1)(2z-4)}{(z^2-4z+3)^2} = -4$

$\therefore Res[f,2] = -4$

Case3 :- $z_0 = 3 \quad , \quad m = 1$

$$\lim_{z\to 3}(z-3)\,\frac{(z^2+1)}{(z-1)(z-2)^2(z-3)}$$

$$\lim_{z\to 3}\frac{(z^2+1)}{(z-1)(z-2)^2} = 5$$

$$\therefore Res[f,3] = 5$$

Dividing Analytic function Method

Definition

To calculate the Residues $f(z)$ on the pole z_0 , we can write the function as

$$f(z) = \frac{p(z)}{q(z)}$$

Such that $p\,,q$ are analytic function and $p(z_0) \neq 0$

Case 1

If $q(z_0) = 0$, $q'(z_0) \neq 0$ then the

$$Res[f\,,z_0] = \frac{p(z_0)}{q'(z_0)}$$

Case 2

If $q(z_0) = 0$, $q'(z_0) = 0$, $q''(z_0) \neq 0$ then

$$Res[f, z_0] = \frac{2p'(z_0)}{q''(z_0)} - \frac{2}{3}\frac{p(z_0)\, q'''(z_0)}{(q''(z_0))^2}$$

Example

Find the Residues for $f(z) = \cot z$

Solution

$f(z) = \cot z = \frac{\cos z}{\sin z}$, $p(z) = \cos z$, $q(z) = \sin z$

$z_0 = n\pi$, $n = \mp 1, \mp 2, \dots, 0$, $n \in z$

$p(n\pi) = \cos(n\pi) = (-1)^n \neq 0$

$Res[f, n\pi] = \frac{p(n\pi)}{q'(n\pi)} = \frac{\cos n\pi}{\cos n\pi} = 1$

Example

calculate the residues at $z = 0$ for the function , $f(z) = \frac{e^{iz}}{\sin z}$

Solution

$z_0 = 0$, $p(z) = e^{iz}$

$p(0) = e^{i0} = 1 \neq 0$

$q(z) = \sin z \rightarrow q(0) = 0$

$q'(z) = \cos z \rightarrow q'(0) = 1$

$Res[f,0] = \frac{1}{1} = 1$

Example

find Residues for $f(z)\frac{1}{z(e^z-1)}$

Solution

$z_0 = 0$

$p(z_0) = 1 \neq 0$

$q(z) = z(e^z - 1) \rightarrow q(0) = 0$

$q'(z) = z(e^z) + e^z - 1 \rightarrow q'(0) = 0$

$q''(z) = ze^z + e^z + e^z \rightarrow q''(0) = 2$

$Res[f\ ,0] = \frac{2p'}{q''} - \frac{2}{3}\frac{p\ \ q'''}{(q'')^2}$

$\frac{2(0)}{2} - \frac{2}{3}\frac{(1\ )\ (3)}{(4)} = -\frac{1}{2}$

$q'''(z) = ze^z + 3e^z \rightarrow q'''(0) = 3$