Ministry of Higher Education and Scientific Research

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COMLEX ANALYSIS

1st Course

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CHAPTER FOUR – ELEMENTARY FUNCTIONS

Exponential Function e^z

let $z \in \mathbb{C}$ then e^z is called the exponential function Variables and can be simplified to

$$e^z = e^{x+iy} = e^x e^{iy} = e^x[\cos y + i \sin y]$$

Properties of Exponential Function

If z_1, z_2, z_3 are complex numbers then

I. e^z is analytic function on $\mathbb{C}$ (entire)

$$e^z = e^{x+iy} = e^x e^{iy} = e^x[\cos y + i \sin y]$$

$$= \underbrace{e^x \cos y}_{u} + i\, \underbrace{e^x \sin y}_{v}$$

$u_x = e^x \cos y \qquad v_y = e^x \cos y$

$u_y = -e^x \sin y \qquad v_x = e^x \sin y$

$u_x = v_y$ & $u_y = -v_x$ [C.R.E]

$\therefore$ fun. $f(z) = e^z$ is analytic , then its entire

II. $e^z \neq 0 , \forall z \in \mathbb{C}$

By Contradiction

let $e^z = 0 \to e^x \cos y + ie^x \sin y = 0 + 0i$

$$\left.\begin{matrix} e^x \cos y = 0 \\ \because e^x \neq 0 \end{matrix}\right\rangle \to \cos y = 0$$

$$\left.\begin{matrix} e^x \sin y = 0 \\ \because e^x \neq 0 \end{matrix}\right\rangle \to \sin y = 0$$

and this is impossible because no angle that the cosine and sine of it's equal to zero. in the Same moment.

III. $|e^z| = e^x$

$$|e^z| = |e^x \cos y + ie^x \sin y|$$

$$= \sqrt{e^{2x}\cos^2 y + e^{2x}\sin^2 y}$$

$$= \sqrt{e^{2x}(\cos^2 y + \sin^2 y)}$$

$$= e^x\sqrt{1} = e^x$$

IV. $e^{z_1}.e^{z_2} = e^{z_1+z_2}$

$$e^{z_1}.e^{z_2} = (e^{x_1}\cos y_1 + ie^{x_1}\sin y_1)(e^{x_2}\cos y_2 + ie^{x_2}\sin y_2)$$

$$= e^{x_1}e^{x_2}\cos y_1\cos y_2 - e^{x_1}e^{x_2}\sin y_1\sin y_2$$

$$+ i(e^{x_1}e^{x_2}\cos y_1\sin y_2 + e^{x_1}e^{x_2}\sin y_1\cos y_2)$$

$$= e^{x_1}e^{x_2}(\cos y_1\cos y_2 - \sin y_1\sin y_2) + ie^{x_1}e^{x_2}(\cos y_1\sin y_2 + $$
$$+ \sin y_1\cos y_2)$$

$$= e^{x_1+x_2}(\cos(y_1+y_2)) + ie^{x_1+x_2}(\sin(y_1+y_2))$$

$$= e^{x_1+x_2}[\cos(y_1+y_2) + i\sin(y_1+y_2)]$$

$$= e^{z_1+z_2}$$

$$\therefore \quad e^{z_1}.e^{z_2} = e^{z_1+z_2}$$

V. $\boldsymbol{\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}}$

$$\frac{e^{z_1}}{e^{z_2}} = \frac{e^{(x_1+iy_1)}}{e^{(x_2+iy_2)}} = \frac{e^{x_1}\ e^{iy_1}}{e^{x_2}\ e^{iy_2}} = \frac{e^{x_1}}{e^{x_2}} \frac{\cos y_1 + i\sin y_1}{\cos y_2 + i\sin y_2}$$

$$\frac{e^{z_1}}{e^{z_2}} = \frac{e^{x_1}}{e^{x_2}} \frac{\cos y_1 + i\sin y_1}{\cos y_2 + i\sin y_2} \times \frac{\cos y_2 - i\sin y_2}{\cos y_2 - i\sin y_2}$$

$$\frac{e^{z_1}}{e^{z_2}} = \frac{e^{x_1}}{e^{x_2}} (\cos y_1 \cos y_2 + \sin y_1 \sin y_2) + i(\sin y_1 \cos y_2 - \sin y_2 \cos y_1)$$

$$\frac{e^{z_1}}{e^{z_2}} = e^{x_1 - x_2}\,[\,\cos(y_1 - y_2) + i\sin(y_1 - y_2)]$$

$$\frac{e^{z_1}}{e^{z_2}} = e^{x_1 - x_2}\ \ e^{i(y_1 - y_2)}$$

$$= e^{z_1 - z_2}$$

VI. for any $\boldsymbol{f(z) = e^z}$ then : $\boldsymbol{r = e^x}$ & $\boldsymbol{\arg(e^z) = y}$

$$e^z = e^{x+iy} = e^x e^{iy}$$

$$= e^x[\cos y + i\sin y]$$

By comparing by polar we conclude

$$r = e^x \ \&\ \arg e^z = y$$

VII. $\boldsymbol{e^z}$ is periodic

we shall prove that $e^{z+2\pi ki} = e^z$ [definition of the periodic function]

$$e^{z+2\pi ki} = e^z\left[e^{2\pi ki}\right] = e^z[\cos 2k\pi + i\sin 2k\pi]$$

$$= e^z[1 + i.0] = e^z[1] = e^z$$

$$e^{z+2\pi ki} = e^z$$

VIII. $e^0 = 1$

IX. $(e^z)^n = e^{zn}$

X. $\overline{(e^z)} = e^{\bar{z}}$

XI. $\frac{1}{e^z} = e^{-z}$

proof : **H.W**

Example :

Find all value of z such that : $e^z = 1+\sqrt{3}i$

Solution

$|e^z| = e^x$

$\left|1+\sqrt{3}i\right| = e^x \rightarrow e^x = \sqrt{1+3} = 2$

$\therefore \quad x = \ln 2$

$\theta = \tan^{-1}\frac{y}{x} \rightarrow \theta = \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3} + 2\pi k$

$\therefore \quad z = x + yi = \ln 2 + i(\frac{\pi}{3} + 2\pi k)\ )$

Example :

Find all value of z such that : $e^z = -2+2i$

Solution

$e^x = |e^z| = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$\therefore \quad x = \ln 2\sqrt{2}$

$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}(\frac{2}{-2})$

$\theta = \frac{\pi}{4} \quad \rightarrow \quad \theta = \frac{3\pi}{4} \quad (\quad (-2,2) \in 2nd\ Q. \quad)$

$\therefore\ z = x + yi = \ln 2\sqrt{2} + i(\frac{3\pi}{4} + 2\pi k)\ )$

Example :

Find all value of z such that : $e^z = 3\sqrt{3} - 9i$

Solution

H.W

Logarithm Function

There are three types of Logarithm Fun. Classified by the type of its base

- **Natural Logarithm written as $\ln z$**
- **Decimal Logarithm Written as $\log_{10} z$ or $\log z$**
- **classical Logarithm Written as $\log_a z$, $a \neq 1$**

Properties of Logarithm function

- $\ln z = \log z$
- $\log z = \log_{10} z$
- $\log_a x = \frac{\log z}{\log a} = \frac{\ln z}{\ln a}$, $a \neq 0$

to calculate of Logarithm Functions for any complex number $z = x + yi$ or $z = re^{i\theta}$ we defined the Logarithim

$$\log z = \ln|z| + i(\theta + 2\pi k)$$

$$\log z = \ln r + i(\theta + 2\pi k)$$

such that $|z| = r$, $\theta = \arg z$

Note:-

the logarithim fun. is an inverse function to the exponential function : $(e^{\log z} = z)$

$$e^{\log z} = e^{\ln|z| + i\, arg\, z}$$

$$= e^{\ln r + i(\theta + 2\pi k)}$$

$$= e^{\ln r} e^{i(\theta + 2\pi k)}$$

$$= r[\cos(\theta + 2\pi k) + i \sin(\theta + 2\pi k)]$$

$$= r[\cos\theta + i \sin\theta] = re^{i\theta} = z$$

Note :

the principle value means that the expression lies in the first period (k=0)

Example :

show that the principle value of Logarithm function is analytic

Solution

$$\log z = \underbrace{\ln r}_{u} + i \underbrace{\theta}_{v}$$

$u = \ln r \quad , \quad v = \theta$

$u_r = \frac{1}{r} \quad , \quad v_r = 0$

$u_\theta = 0 \quad , \quad v_\theta = 1$

$v_\theta = r u_r \;,\; u_\theta = -r v_r$

∴ C.R.E. in polar are satisfied then principle value of $\log z$ is analytic

Properties of logarithmic function

1. $\mathbf{\log z_1 z_2 = \log z_1 + \log z_2}$

$$\log z_1 z_2 = \log|z_1 z_2| + i\arg(z_1 z_2)$$
$$= \log|z_1||z_2| + i(\arg z_1 + \arg z_2)$$
$$= \log|z_1| + \log|z_2| + i\arg z_1 + i\arg z_2$$
$$= (\log|z_1| + i\arg z_1) + (\log|z_2| + i\arg z_2)$$
$$= \log z_1 + \log z_2$$

2. $\boldsymbol{log\frac{z_1}{z_2} = log\, z_1 - log\, z_2}$

$$log\frac{z_1}{z_2} = log\left|\frac{z_1}{z_2}\right| + i\,arg(\frac{z_1}{z_2})$$
$$= log\frac{|z_1|}{|z_2|} + i(arg\, z_1 - arg\, z_2)$$
$$= log|z_1| - log|z_2| + i\,arg\, z_1 - i\,arg\, z_2$$
$$= log|z_1| + i\,arg\, z_1 - [log|z_2| + i\,arg\, z_2]$$
$$= log\, z_1 - log\, z_2$$

3. $\boldsymbol{\log e^z = z + 2\pi k i \quad , \; k \in Z}$

$$\log e^z = \ln|e^z| + i\,arg e^z$$
$$= \ln e^x + i(\theta + 2\pi k)$$
$$= x + i(y + 2\pi k)$$
$$= x + iy + 2\pi k i = z + 2\pi k i$$