4. $\log z^{\frac{1}{n}} = \frac{1}{n}\log z \quad , \ n = 1,2,3,\ldots$

$$\log z^{\frac{1}{n}} = \log \sqrt[n]{z} = \log[r^{\frac{1}{n}}\, e^{i\left(\frac{\theta+2\pi k}{n}\right)}]$$

$$= \log r^{\frac{1}{n}} + \log\ e^{i\left(\frac{\theta+2\pi k}{n}\right)}$$

$$= \frac{1}{n}\ln r + i\left(\frac{\theta+2\pi k}{n}\right)$$

$$= \frac{1}{n}\left(\ln r + i(\theta + 2\pi k)\right)$$

$$= \frac{1}{n}\log z$$

5. $\log\frac{1}{z} = -\log z \quad , \ z \neq 0$

$$\log\frac{1}{z} = \ln\left|\frac{1}{z}\right| + i\ \arg\left(\frac{1}{z}\right)$$

$$= \ln\frac{1}{|z|} - i\arg(z)$$

$$= -\ln|z| - i\arg(z)$$

$$= -(\ln|z| + i\arg(z))$$

$$= -\log z$$

Example :

Find $\log(-2-2i)$

Solution

$$|z| = \sqrt{4+4} = 2\sqrt{2}$$

$$\theta = \tan^{-1}\frac{-2}{-2} = 1 \rightarrow \theta = \frac{\pi}{4} \xrightarrow{3rd.Q.} \theta = \frac{5\pi}{4}$$

$\therefore \log z = \ln 2\sqrt{2} + i\left(\frac{5\pi}{4} + 2\pi k\right) \quad , k = 0, \mp 1, \mp 2, \ldots.$

Important values :-

$\log 1 = 0$

$\log i = \frac{\pi}{2} i$

$\log(-1) = \pi i$

$\log(-i) = \frac{3\pi}{2} i = -\frac{\pi}{2} i$

Note : $\log z^k \neq k \log z$

Example :

is $\log(i)^2 = 2\log i$ **?**

Solution

L.H.S

$\log(i)^2 = \log(-1)$

$= \ln|-1| + i(\theta + 2\pi k)$

$= \ln 1 + i(\pi + 2\pi k)$

$= i\pi(1 + 2k)$

R.H.S

$2\log i = 2[\ln|i| + i(\theta + 2\pi k)]$

$= 2[\ln 1 + i\left(\frac{\pi}{2} + 2\pi k\right)]$

$= 2[0 + i\left(\frac{\pi}{2} + 2\pi k\right)]$

$= i(\pi + 4\pi k) = i\pi(1 + 4k)$

We note that $\log(i)^2 \neq 2\log i$

Example :

is $\log(-i)^4 = 4\log(-i)$? if No ,find the value of k that the expression are equal

Solution

L.H.S

$\log(-i)^4 = \log(1)$

$= \ln|1| + i(\theta + 2\pi k)$

$= (0 + i(0 + 2\pi k)) = 2\pi k i$

R.H.S

$4\log(-i) = 4[\ln|-i| + i(\theta + 2\pi k)]$

$= 4\left[\ln 1 + i\left(\frac{3\pi}{2} + 2\pi k\right)\right]$

$= 4[0 + i\left(\frac{3\pi}{2} + 2\pi k\right)]$

$= (6\pi + 8\pi k)i$

$\log(-i)^4 \neq 4\log(-i)$

to find the k values that satisfy equality

$2\pi k = 6\pi + 8\pi k \qquad \div 2\pi$

$k = 3 + 4k$

$3k = -3 \rightarrow k = -1$

Example :

is $log(1+i)^2 = 2\,log(1+i)$?

Solution

L.H.S

$$\log(1+i)^2 = \log(2i)$$

$$= \log|2i| + i(\theta + 2\pi k)$$

$$= \log|2i| + i\left(\frac{\pi}{2} + 2\pi k\right)$$

$$= \ln 2 + i\left(\frac{\pi}{2} + 2\pi k\right)$$

$$= \ln 2 + i\pi\left(\frac{1}{2} + 2k\right)$$

R.H.S

$$2\log(1+i) = 2[\ln|1+i| + i(\theta + 2\pi k)]$$

$$= 2\left[\ln\sqrt{2} + i\left(\frac{\pi}{4} + 2\pi k\right)\right]$$

$$= 2\ln\sqrt{2} + i\left(\frac{\pi}{2} + 4\pi k\right)$$

$$= \ln 2 + i\pi\left(\frac{1}{2} + 2k\right)$$

$$\therefore \log(1+i)^2 \neq 2\log(1+i)$$

Example :

1- is $\log(1-i)^4 = 4\log(1-i)$?
2- is $\log(2i)^5 = 5\log(2i)$?
3- find $\log(1-\sqrt{3})$

H.W

Example :

Show that $\log(-ie) = 1 - \frac{\pi}{2} i$

Solution

$\log(-ie) = \ln|-ie| + i\theta$

$= \ln|-i||e| + i(\frac{-\pi}{2})$

$= \ln e + i\left(\frac{-\pi}{2}\right) = 1 - \frac{\pi}{2} i$

Example :

Show that the function $\log(z - i)$ is analytic every where except on the half line $y = 1$ and $x \le 0$

Solution

$z - i = 0 \rightarrow z = i$ (Singular point)

$x + yi = i$

$x = 0$ and $y = 1$

Real no. is defined on $x > 0$ and imaginary in all values $y \neq 1$

$\therefore$ the fun. is analytic except $y = 1$, $x \le 0$

Example :

Show that the function $\frac{\log(z+4)}{z^2+i}$ is analytic every where except at the points $\mp \frac{(1-i)}{\sqrt{2}}$ and the half line $x \le 4$, $y = 0$

Solution

$z + 4 = 0 \rightarrow z = -4$

$x + yi = -4 + 0i$

$x = -4 \quad , \quad y = 0$

$z^2 + i = 0 \ \rightarrow z^2 = -i \ \rightarrow z = \sqrt{-i}$

$r = \sqrt{0+1} = 1$

$\theta = \tan^{-1}\frac{y}{x} \rightarrow \theta = \frac{3\pi}{2}$

$z_n = (r)^{\frac{1}{n}}\left[\cos(\frac{\theta+2\pi k}{n}) + i\sin(\frac{\theta+2\pi k}{n})\right]$

$z_0 = (r)^{\frac{1}{2}}\left[\cos(\frac{\frac{3\pi}{2}+2\pi(0)}{2}) + i\sin(\frac{\frac{3\pi}{2}+2\pi(0)}{2})\right]$

$z_0 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -\cos\frac{\pi}{4} + \sin\frac{\pi}{4}$

$= \frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \frac{(-1+i)}{\sqrt{2}}$

$z_1 = (r)^{\frac{1}{2}}\left[\cos(\frac{\frac{3\pi}{2}+2\pi}{2}) + i\sin(\frac{\frac{3\pi}{2}+2\pi}{2})\right]$

$z_1 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \cos\frac{\pi}{4} - \sin\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} = \frac{(1-i)}{\sqrt{2}}$

$\therefore f(z)$ is analytic every where except $\mp\frac{(1-i)}{\sqrt{2}}$ and $y = 0 \quad , \ x \leq 4$

Complex fun. Of formula Z^c

$z^c = e^{\log z^c} = e^{c\log z}$

$$Z^c = e^{c\,[\ln(r)+i(\theta+2\pi k)]}$$

And the principle value Z^c is

$$Z^c = e^{c\,[\ln(r)+i(\theta)]}$$

Example :

Find the value of :

I. $(-i)^i$

$(-i)^i = e^{\log (-i)^i} = e^{i \log(-i)}$

$= e^{i\left[\ln 1 + i\frac{3\pi}{2}\right]} = e^{-\frac{3\pi}{2}} = e^{\frac{\pi}{2}}$

II. $(1+i)^i$

$(1+i)^i = e^{\log (1+i)^i} = e^{i \log(1+i)}$

$= e^{i\left[\log \sqrt{2} + i\left(\frac{\pi}{4} + 2\pi k\right)\right]}$

$P(z^c) = e^{i\left[\log \sqrt{2} + i\frac{\pi}{4}\right]}$

III. $(1-i)^{2-i}$

$(1-i)^{2-i} = e^{\log (1-i)^{2-i}} = e^{(2-i) \log(1-i)}$

$= e^{(2-i)\left[\log \sqrt{2} + i\left(\frac{7\pi}{4} + 2\pi k\right)\right]}$

$P(1+i)^{2-i} = e^{(2-i)\left[\log \sqrt{2} + \frac{7\pi}{4} i\right]}$

IV. $(2-3i)^{(-\sqrt{3}-i)}$

H.W