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complex analysis

2nd Course

Chapter 4

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Chapter Four

Improper Integral

It is possible to calculate the values of real integrals over finite periods or over infinite periods using complex integrals by taking the complex function corresponding to the real function and choosing a simple closed curve as the path for the complex integration. Below are some notes that help in finding the values of integrals and some examples that illustrate this when finding the integral :

$$\int_{-\infty}^{\infty} f(x)\,dx$$

If the function $f(z)$ satisfies the conditions , this integral of the function can be found if $f(z)$ is analytic in the upper half of the plane, except for the points that are poles of the function.

The residue theorem can be used to calculate some defective real integrals. The defective integral is :

$$\int_{-\infty}^{\infty} f(x)\,dx$$

Asymptotically where f is a continuous function for all values of x if the limit

$$\lim_{R\to\infty} \int_{-R}^{0} f(x)\,dx + \int_{0}^{R} f(x)\,dx \qquad (*)$$

exists for both of these integrals, and the last equation can be rewritten as follows

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

The last integral is called the Cauchy Principle value of the integral

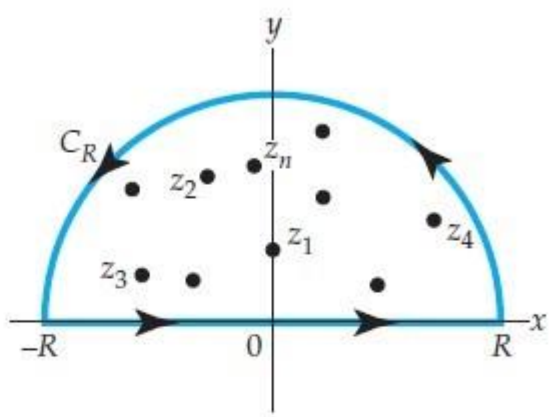
The limit on the right side exists. If the integral (*) is convergent, then its value is the principal value of Cauchy.

1) if the function $f(x) = x$ then the value of integral is zero

2) if the function is an even function i.e $f(-x) = f(x)$, then

$$\int_{-R}^{0} f(x)\,dx = \int_{0}^{R} f(x)\,dx = \frac{1}{2}\int_{-R}^{R} f(x)\,dx$$

3) if the function is fractional function , $f(x) = \frac{p(x)}{q(x)}$ such that $p(x)\,,q(x)$ are polynomials and no common factor between them , the degree of $q(x)$ is greater than the degree of $p(x)$ at least by two , if this conditions satisfies then the integral is convergence and can be calculated by Residues theorem.



Def:- (Cauchy Principle value PV)

let $f(x)$ be a centinous real valued function for all x , the cauchy Principle value (P.V) of the integral $\int_{-\infty}^{\infty} f(x)dx$ is defined by

$$\text{P.V} \quad \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx$$

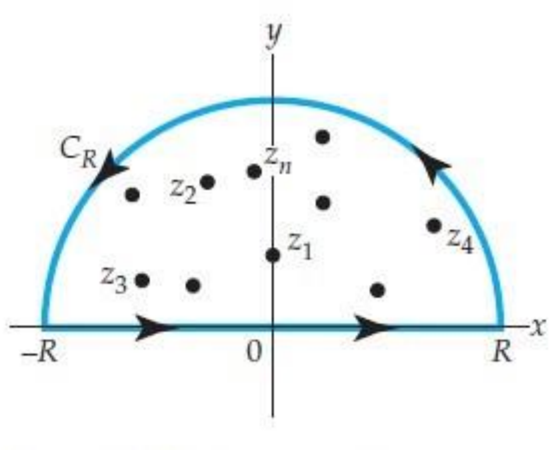
provided the limit exists

Theorem

Let $f(x) = \frac{p(x)}{Q(x)}$ where P and Q are polynomial of degree m and n respectively if $Q(x) \neq 0$ for all real x and $n \geq m+2$ then :

$$\text{P.V} \quad \int_{-\infty}^{\infty} \frac{p(x)}{Q(x)} dx = 2\pi i \sum_{i=1}^{k} Res\left(\frac{P(zi)}{Q(zi)}\right)$$

Where $z_1, z_2, \dots, z_k$ and the poles of $\frac{P}{Q}$ that lie in the upper half plane the situation is illustrated in the figure:



Such that the pales of $\frac{P}{Q}$ that lie in the upper half plane, So we can find a real number R, such that the poles all lie inside the counter C which consist the segment of the $x-axis$ together with the upper semi circle C_R of radius R shown in a bove figure

Example

Find the P.V of the $\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx$

Solution:

$$\text{P.V } \int_{-\infty}^{\infty}\frac{1}{x^2+1}dx = \lim_{R\to\infty}\int_{-R}^{R}\frac{1}{1+z^2}dz$$

$z = \mp i$ is a simple pole ; $z = i \in C_R$

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{1}{z^2+1}dz = 2\pi i(Res\,[f(z), i])$$

$$= 2\pi i[\lim_{z\to i}(z-i)\frac{1}{(z-i)(z+i)}$$

$$= 2\pi i\left(\frac{1}{2i}\right) = \pi$$

Example

Evaluate $\int_{0}^{\infty}\frac{x^2}{(x^2+16)^2}dx$

H.W

Example

Evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)}$

Solution:

We write the integrand as

$$f(z) = \frac{1}{(z^2+1)(z^2+4)} = \frac{1}{(z+i)(z-i)(z+2i)(z-2i)}$$

We see that $f(z)$ has a simple poles at the point i , $2i$ in the upper half plane ,Computing the Residues

$$Res[f,i] = \lim_{z\to i}(z-i)\frac{1}{(z+i)(z-i)(z^2+4)} = \frac{-i}{6}$$

$$Res[f,2i] = lim_{z\to i}(z-2i)\frac{1}{(z^2+i)(z+2i)(z-2i)} = \frac{-i}{12}$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)} = 2\pi i\left(\frac{-i}{6} + \frac{i}{12}\right) = \frac{\pi}{6}$$

Example

Evaluate $\int_{-\infty}^{\infty} \frac{3x^2+2}{x^2+2x+2}\, dx$

H.W