**Example**

**Evaluate** $\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^3}$

**Solution:**

The integrand $\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^3}$ have a pole from the 3rd order at the point $2i$ which is only singularity of $f$ in the upper half plane computing the Residues

$\lim_{z\to 2i}(z-2i)^3 \frac{1}{(z^2+4)^3} = \lim_{z\to 2i}(z-2i)^3 \frac{1}{(z-2i)^3(z+2i)^3} = \frac{i}{64} \neq 0 \quad , \quad m=3$

$Res[f,2i] = \frac{1}{2!}\lim_{z\to 2i} \frac{d^2}{dz^2} \frac{1}{(z+2i)^3}$

$$f(z) = (z+2i)^{-3} \xrightarrow{\prime} f'(z) = -3(z+2i)^{-4} \xrightarrow{\prime\prime} f''(z) = 12(z+2i)^{-5}$$

$Res[f,2i] = \frac{1}{2}\lim_{z\to 2i} \frac{12}{(z+2i)^5}$

$$= \frac{1}{2} \frac{12}{(4i)^5} = \frac{-3i}{512}$$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{(x^2+4)^3} = 2\pi i\left(\frac{-3i}{512}\right) = \frac{3}{256}\pi$$

**Note:-**

if $f(x): -\infty < x < \infty$ is an even function one where $f(-x) = f(x)$ for all $x$ and assume that the cauchy principle value exists , the symmetry of the graph of $y = f(x)$ with respect to the $y-axis$ tells us that

$\int_{-R}^{R} f(x)dx = \int_{-R}^{0} f(x)dx + \int_{0}^{R} f(x)dx$

$\because \int_{-R}^{0} f(x)dx = \int_{0}^{R} f(x)dx$ then

$\int_{-R}^{R} f(x)dx = 2\int_{0}^{R} f(x)dx = 2\int_{-R}^{0} f(x)dx$

**Example**

**Evaluate** $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+2x+2)} dx$

**Solution:**

$f(z) = \frac{z^2}{(z+i)(z-i)(z+1-i)(z+1+i)}$

The function having poles : $i, -1+i$ lies in the upper half plane

$$Res[f,i] = \lim_{z\to i}(z-i)\frac{z^2}{(z+i)(z-i)(z^2+2z+2)}$$

$$Res[f,i] = \frac{-1}{(2i)(1+2i)} = \frac{1}{4-2i} = \frac{4+2i}{20}$$

$$Res[f,-1+i] = \lim_{z\to -1+i}(z+1-i)\frac{z^2}{(z^2+1)(z+1-i)(z+1+i)}$$

$$= \frac{(-1+i)^2}{((-1+i)^2+1)(-1+i+1+i)} = \frac{-2i}{(1-2i)(2i)}$$

$$Res[f,-1+i] = \frac{-1}{1-2i} = \frac{-1-2i}{5}$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+2x+2)} = 2\pi i\left(\frac{4+2i}{20} - \frac{1+2i}{5}\right) = \frac{3\pi}{5}$$

**Example**

**Evaluate $\int_0^\infty \frac{dx}{(x^2+1)^2}$**

**Solution:**

$$\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{(x^2+1)^2}$$

$$\int_{-\infty}^\infty \frac{dx}{(x^2+1)^2} = \lim_{R\to\infty}\int_{-R}^{R} \frac{dz}{(z^2+1)^2}$$

$z = i$ is The poles from 2nd order

$$\emptyset(z) = \lim_{z\to i}(z-i)^2 \frac{1}{(z-i)^2(z+i)^2} = \frac{-1}{4} \neq 0$$

$$Res[f,i] = \lim_{z\to i} \frac{d}{dz}(z-i)^2 \frac{1}{(z-i)^2(z+i)^2}$$

$$Res[f,i] = \lim_{z\to i} \frac{d}{dz}\frac{1}{(z+i)^2}$$

$$Res[f,i] = \lim_{z\to i} \frac{-2}{(z+i)^3} = \frac{-2}{-8i} = \frac{-i}{4}$$

$$\therefore \int_{-\infty}^\infty \frac{1}{(x^2+1)^2}\,dx = 2\pi i\left[\frac{-i}{4}\right] = \frac{\pi}{2}$$

$$\therefore \int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$$

**Example**

**Evaluate** $\int_{-\infty}^{\infty} \frac{x+3}{(x^2+9)} dx$

**H.W**

**Example**

**Evaluate** $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$

**Solution:**

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \lim_{R\to\infty} \int_{-\infty}^{\infty} \frac{dz}{z^2+2z+2}$$

The function $f(z)$ have a simple pole $-1+i$ lies in the upper half plane

$$Res[f,-1+i] = \lim_{z\to -1+i} (z+1-i)\frac{1}{(z+1-i)(z+1+i)}$$

$$Res[f,-1+i] = \frac{1}{2i}$$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{x^2+2x+2} dx = 2\pi i\left(\frac{1}{2i}\right) = \pi$$

**Example**

**calculate** $\int_0^\infty \frac{\ln(x+i)}{x^2+1}$

**Solution:**

$$\int_0^\infty \frac{\ln(x+i)}{x^2+1}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\ln(x+i)}{x^2+1}dx$$

$f(z) = \frac{\ln(z+i)}{z^2+1}$ having simple pole $z = i$ in the upper half plane , calculate Residues

$$Res[f,i] = \lim_{z\to i}(z-i)\frac{\ln(z+i)}{(z-i)(z+i)}$$

$$= \frac{\ln(2i)}{2i}$$

$$\therefore \int_{-\infty}^\infty \frac{\ln(z+i)}{z^2+1}dz = 2\pi i\, Res[f,i] = 2\pi i\frac{\ln 2i}{2i} = \pi \ln 2i$$

$$\therefore \int_0^\infty \frac{\ln(z+i)}{z^2+1}dz = \frac{1}{2}(\pi \ln 2i)$$

$$= \frac{\pi}{2}\ln 2i = \frac{\pi}{2}\left(\ln 2 + i\frac{\pi}{2}\right)$$

**Example**

**calculate** $\int_0^\infty \frac{dx}{x^4+16}dx$

**H.W**