Ministry of Higher Education and Scientific Research
University of Mosul
College of Computer Science and Mathematics
Mathematics Department

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COMLEX ANALYSIS

1st Course

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Dr. MOHAMMED SABAH AL TAEE

CHAPTER FIVE

TRIGONOMETRIC FUNCTIONS

Trigonometric function

the Trigonometric Function. $\sin z$ and $\cos z$ are entire function i.e that is analytic on all points because its compose of the function e^{iz}, e^{-iz}

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad , \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sinh z = \frac{e^{z} - e^{-z}}{2} \quad , \quad \cosh z = \frac{e^{z} + e^{-z}}{2}$$

NOTE:-

$\cos(ix) = \cosh x$

$$\cos(ix) = \frac{e^{i(ix)} + e^{-i(ix)}}{2} = \frac{e^{x} + e^{-x}}{2} = \cosh(x)$$

$\sin(ix) = i \sinh x$

H.W

1) $\cos z = \cos x \cosh y - i \sin x \sinh y$

Proof :-

$$\cos z = \cos(x + yi) = \frac{1}{2}\{e^{iz} + e^{-iz}\}$$

$$= \frac{1}{2}\{e^{i(x+yi)} + e^{-i(x+yi)}\}$$

$$= \frac{1}{2}\{e^{ix}e^{-y} + e^{-ix}e^{y}\}$$

$$= \frac{1}{2}[e^{-y}(\cos x + i \sin x) + e^{y}(\cos x - i \sin x)]$$

$= \frac{1}{2} e^{-y} \cos x + \frac{1}{2} i e^{-y} \sin x + \frac{1}{2} e^{y} \cos x - \frac{1}{2} i\, e^{y} \sin x$

$= \frac{1}{2} \cos x\, (e^{y} + e^{-y}) - \frac{1}{2} i \sin x\, (e^{y} + e^{-y})$

$= \cos x \frac{(e^{y}+e^{-y})}{2} - i \sin x \frac{(e^{y}+e^{-y})}{2}$

$\boldsymbol{\cos z = \cos x \cosh y - i \sin x \sinh y}$

2) $\boldsymbol{sin\, z = sin\, x\ \ cosh\, y + i\, cos\, x\ \ cosh\, y}$

Proof :-

$sin\, z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = \frac{1}{2i}\left[e^{i(x+yi)} - e^{-i(x+yi)}\right]$

$= \frac{1}{2i}\left[e^{ix} e^{-y} - e^{-ix} e^{y}\right]$

$= \frac{1}{2i}\left[e^{-y}(\cos x + i \sin x) - e^{y}(\cos x - i \sin x)\right]$

$= \frac{1}{2i} e^{-y} \cos x + \frac{1}{2} e^{-y} \sin x - \frac{1}{2i} e^{y} \cos x + \frac{1}{2} e^{y} \sin x$

$= \frac{1}{2} \sin x\, (e^{y} + e^{-y}) - \frac{1}{2i} \cos x\, (e^{y} - e^{-y})$

$= \sin x \left(\frac{e^{y} + e^{-y}}{2}\right) + i \cos x \left(\frac{e^{y} - e^{-y}}{2}\right)$

$\boldsymbol{\sin z = \sin x \cosh y + i \cos x \sinh y}$

3) $\boldsymbol{tan\, z = \frac{sin\, 2x}{cos\, 2x\, cosh 2y} + \frac{sinh\, 2y}{cos\, 2x\, cosh 2y}}$

Proof :-

$tan\, z = \frac{\sin z}{\cos z} = \frac{\sin(x+yi)}{\cos(x+yi)} \times \frac{\cos(x-yi)}{\cos(x-yi)}$

$= \frac{\sin(x+yi)\cos(x-yi)}{\cos(x+yi)\cos(x-yi)}$

$\boldsymbol{\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))}$

$\boldsymbol{\cos A \sin B = \frac{1}{2}(\cos(A+B) + \cos(A-B))}$

$$= \frac{\sin(x+yi)\cos(x-yi)}{\cos(x+yi)\cos(x-yi)}$$

$$= \frac{\frac{1}{2}[\sin(x+yi+x-yi)+\sin(x+yi-x+yi)]}{\frac{1}{2}[\cos(x+yi+x-yi)+\cos(x+yi-x+yi)]}$$

$$= \frac{[\sin(2x)+\sin(2yi)]}{[\cos(2x)+\cos(2yi)]} = \frac{\sin(2x)+i\sinh(2y)}{\cos(2x)+\cosh(2y)}$$

$$= \underbrace{\frac{\sin(2x)}{\cos(2x)+\cosh(2y)}}_{u(x,y)} + i\underbrace{\frac{\sinh(2y)}{\cos(2x)+\cosh(2y)}}_{v(x,y)}$$

Example

prove that $f(z) = \sin z$ interior function

Solution

$\sin z = \underbrace{\sin x \cosh y}_{u} + i\,\underbrace{\cos x \sinh y}_{v}$

$u_x = \cos x \cosh y$, $v_y = \cos x \cosh y$

$u_y = \sin x \sinh y$, $v_x = -\sin x \sinh y$

C.R.E are satisfied

since the partial derivatives are continuous and differentiable , then

$\therefore f(z)$ is enterior

Example

is $f(z) = \cos z$ interior function ?

H.W

Example

the zeros of $\sin z$ is real (solve $sin\ z = 0$)

Solution

$\sin z = 0$

$\frac{e^{iz} - e^{-iz}}{2i} = 0 \qquad \times 2i$

$e^{iz} - e^{-iz} = 0 \qquad \times e^{iz}$

$e^{2iz} - 1 = 0$

$\cos 2z + i \sin 2z = 1$

$\cos 2z = 1$

$2z = 2\pi k \rightarrow z = \pi k$

$\therefore\ z$ is real

Example

prove that $\overline{(\sin z)} = \sin \bar{z}$

H.W

Example

The function $\sin \bar{z}$ are not analytic

Solution

$\sin z = \sin x \cosh y + i \cos x \sinh y$

$\sin \bar{z} = \underbrace{\sin x \cosh y}_{u} - i \underbrace{\cos x \sinh y}_{v}$

$u_x = \cos x \cosh y \quad , \quad v_y = -\cos x \cosh y$

$u_y = \sin x \sinh y \quad , \quad v_x = \sin x \sinh y$

C.R.E are not satisfied

$\therefore \sin \bar{z}$ not analytic

Important Rules

1. $e^{iz} = \cos z + i\sin z$, $e^{-iz} = \cos z - i\sin z$
2. $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$, $\cos z = \frac{e^{iz}+e^{-iz}}{2}$

   $\tan z = \frac{\sin z}{\cos z}$, $\cot z = \frac{\cos z}{\sin z}$, $\sec z = \frac{1}{\cos z}$, $\csc z = \frac{1}{\sin z}$
3. $\sin(iy) = i\sinh y$, $\cos(iy) = \cosh y$
4. $\sin(z_1 \mp z_2) = \sin z_1 \cos z_2 \mp \sin z_2 \cos z_1$
5. $\cos(z_1 \mp z_2) = \cos z_1 \cos z_2 \pm \sin z_1 \sin z_2$
6. $\cos 2z = \cos^2 z - \sin^2 z$, $\sin 2z = 2\sin z \cos z$
7. $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$
8. $\sin^2 z + \cos^2 z = 1$, $\cosh^2 z - \sinh^2 z = 1$
9. $\sin(z + 2\pi k) = \sin z$

   $\cos(z + 2\pi k) = \cos z$

   $\sin(z + \pi) = -\sin z$

   $\cos(z + \pi) = -\cos z$

   $\tan(z + \pi) = \tan z$
10. $\sin(\frac{\pi}{2} - z) = \cos z$
11. $|\sin z|^2 = \sin^2 x + \sinh^2 y$
12. $|\cos z|^2 = \cos^2 x + \sinh^2 y$
13. $\sin z = 0 \leftrightarrow z = n\pi$, $\cos z = 0 \leftrightarrow z = (n + \frac{1}{2})\pi$

Example

prove that : $|\sin z|^2 = \sin^2 x + \sinh^2 y$

Solution

$sin\, z = sin\, x\, cosh\, y + i\, cos\, x\, sinh\, y$

$|sin\, z| = \sqrt{sin^2 x\, cosh^2 y + cos^2 x\, sinh^2 y}$

$|sin\, z|^2 = sin^2 x\, cosh^2 y + cos^2 x\, sinh^2 y$

$= sin^2 x\,(1 + sinh^2 y) + cos^2 x\, sinh^2 y$

$= sin^2 x + sin^2 x\, sinh^2 y + cos^2 x\, sinh^2 y$

$= sin^2 x + sinh^2 y\,(sin^2 x + cos^2 x)$

$= sin^2 x + sinh^2 y$

Example

prove that : $|cos\, z|^2 = cos^2 x + sinh^2 y$

Solution

$\cos z = \cos x \cosh y - i \sin x \sinh y$

$|\cos z| = \sqrt{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}$

$|\cos z|^2 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$

$= \cos^2 x\,(1 + \sinh^2 y) + \sin^2 x \sinh^2 y$

$= \cos^2 x + \cos^2 x \sinh^2 y + \sin^2 x \sinh^2 y$

$= \cos^2 x + \sinh^2 y\,(\sin^2 x + \cos^2 x)$

$= \cos^2 x + \sinh^2 y$

Example

prove that : $|\cos z|^2 + |\sin z|^2 \geq 1$

Solution

$$|\cos z|^2 + |\sin z|^2 = \cos^2 x + \sinh^2 y + \cos^2 x + \sinh^2 y$$

$$= 1 + 2\sinh^2 y \geq 1 \quad , \quad \because \ \sinh^2 y \geq 0$$

Example

prove that : $|\sinh y| \leq |\sin z| \leq |\cosh y|$

Solution

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$

$$|\sin z|^2 \geq \sinh^2 y \qquad \ldots\ldots\ldots (1)$$

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$

$$|\sin z|^2 = \sin^2 x + \cosh^2 y - 1$$

$$= \sin^2 x - 1 + \cosh^2 y$$

$$= -\cos^2 x + \cosh^2 y$$

$$= \cosh^2 y - \cos^2 x$$

$$|\sin z|^2 \leq |\cosh y|^2 \qquad \ldots\ldots\ldots (2)$$

From (1) & (2) we get

$$\sinh^2 y \leq |\sin z|^2 \leq \cosh^2 y$$

$$\sinh y \leq |\sin z| \leq \cosh y$$