Example

prove that : $|\sin z| \geq |\sin x|$

Solution

$|\sin z|^2 = \sin^2 x + \sinh^2 y$

$|\sin z|^2 \geq \sin^2 x$

$|\sin z| \geq |\sin x|$

Example

prove that : $|\cos z| \geq |\cos x|$

H.W

Example

solve the equation $\sin z = \cosh y$

Solution

$\sin z = \cosh y$

$\sin x \cosh y + i \cos x \sinh y = \cosh y$

$\sin x \cosh y = \cosh y \qquad \dots\dots\dots(1)$

$\cos x \ \sinh y = 0 \qquad \dots\dots\dots(2)$

From (2) we get

Either $\cos x = 0 \to x = (2n+1)\frac{\pi}{2} \qquad , n = 0, \mp 1, \dots$

$\sinh y = 0 \rightarrow \; y = 0$

$\therefore \;\; z = (2n+1)\frac{\pi}{2} + 0i$

Note :- may be find x from (1)

Derivative of Trig. Fun.

1- $\frac{d}{dz}\sin z = \cos z$

2- $\frac{d}{dz}\cos z = -\sin z$

3- $\frac{d}{dz}\tan z = \sec^2 z$

4- $\frac{d}{dz}\cot z = -\csc^2 z$

5- $\frac{d}{dz}\sec z = \sec z \tan z$

6- $\frac{d}{dz}\csc z = -\csc z \cot z$

Proof

1) $(\sin z)' = \left(\frac{e^{iz}-e^{-iz}}{2i}\right)'$

$$\frac{ie^{iz} - ie^{-iz}}{2i} = \frac{i\left(e^{iz} - e^{-iz}\right)}{2i} = \cos z$$

2-6) **H.W**

Hyperbolic Trigonometric Function

(Rules)

1. $\sinh z = \frac{e^z - e^{-z}}{2}$, $\cosh z = \frac{e^z + e^{-z}}{2}$
2. $\sinh(z_1 \mp z_2) = \sinh z_1 \cosh z_2 \mp \sinh z_2 \cosh z_1$
3. $\cosh(z_1 \mp z_2) = \cosh z_1 \cosh z_2 \mp \sinh z_1 \sinh z_2$
4. $\sinh(-z) = -\sinh z$, $\cosh(-z) = \cosh z$
5. $\sinh(iz) = i \sin z$, $\cosh(iz) = \cos(z)$

   $\sin(iz) = i \sinh z$, $\cos(iz) = \cosh(z)$

6. $\sinh z = \sinh x \cos y + i \cosh x \cos y$
7. $\cosh z = \cosh x \cos y - i \sinh x \sin y$
8. $|\sin z|^2 = \sinh^2 x + \sin^2 y$
9. $|\cos z|^2 = \sinh^2 x + \cos^2 y$
10. $(\sinh z)' = \cosh z$, $(\cosh z)' = \sinh z$

    $(\tanh z)' = \operatorname{sech}^2 z$, $(\coth z) = -\operatorname{csch}^2 z$

    $(\operatorname{sech} z)' = -\operatorname{sech} z \tanh z$

    $(\operatorname{csch} z)' = -\operatorname{csch} z \coth z$

Proof : H.W

Inverse Trigonometric function

1. $\cos^{-1} z = \frac{1}{i} \log\left(z \mp \sqrt{z^2 - 1}\right)$
2. $\sin^{-1} z = \frac{1}{i} \log\left(iz + \sqrt{z^2 - 1}\right)$
3. $\tan^{-1} z = \frac{i}{2} \log\left[\frac{i+z}{i-z}\right]$, $z \neq \mp i$
4. $\cot^{-1} z = \frac{1}{2i} \log\left[\frac{z+i}{z-i}\right]$, $z \neq \mp i$
5. $\sec^{-1} z = \frac{1}{i} \log\left[\frac{1+\sqrt{1-z^2}}{z}\right]$
6. $\csc^{-1} z = \frac{1}{i} \log\left[\frac{1+\sqrt{z^2-1}}{z}\right]$

Proof :

1) Let $w = \cos^{-1} z \to z = \cos w$

$$z = \frac{e^{iw}+e^{-iw}}{2} \qquad \times 2e^{iw}$$

$$e^{2iw} - 2ze^{iw} + 1 = 0$$

$$e^{iw} = \frac{2z \mp \sqrt{4z^2-4(1)(1)}}{2(1)}$$

$$= \frac{2z \mp \sqrt{4z^2-4}}{2}$$

$$= \frac{2z \mp 2\sqrt{z^2-1}}{2(1)} \to e^{iw} = z \pm \sqrt{z^2-1}$$

$$iw = \log\left(z \mp \sqrt{z^2-1}\right)$$

$$w = \frac{1}{i}\log\left(z \mp \sqrt{z^2-1}\right)$$

$$\therefore \cos^{-1} z = \frac{1}{i}\log\left(z \mp \sqrt{z^2-1}\right)$$

3) Let $w = \tan^{-1} z \to z = \tan w$

$$z = \frac{1}{i}\frac{e^{iw}-e^{-iw}}{e^{iw}+e^{-iw}}$$

$$iz(e^{iw}+e^{-iw}) = e^{iw} - e^{-iw} \quad , \quad \times e^{iw}$$

$$iz(e^{i2w}+1) = e^{i2w} - 1$$

$$ize^{i2w} + iz - e^{i2w} + 1 = 0$$

$$e^{i2w}(iz-1) = -1 - iz$$

$$e^{i2w} = \frac{-(1+iz)}{(iz-1)} = \frac{i^2(1+iz)}{i(z+i)} = \frac{i-z}{z+i}$$

$$2wi = log\frac{i-z}{i+z}$$

$$w = \frac{1}{2i}log\frac{i-z}{i+z} = \frac{-i}{2}log\frac{i-z}{i+z} = \frac{i}{2}log(\frac{i-z}{i+z})^{-1} = \frac{i}{2}log(\frac{i+z}{i-z})$$

$$\therefore \quad tan^{-1} z = \frac{i}{2} log\ \frac{i+z}{i-z}$$

Proof of 2,4,5,6 are H.W

Derivative of Inv. Trig. Function

1. $(\cos^{-1} z)' = \frac{-1}{\sqrt{1-z^2}}$
2. $(\sin^{-1} z)' = \frac{1}{\sqrt{1-z^2}}$
3. $(\tan^{-1} z)' = \frac{1}{1+z^2}$, $z \neq \mp i$
4. $(\cot^{-1} z)' = \frac{-1}{1+z^2}$, $z \neq \mp i$
5. $(\sec^{-1} z)' = \frac{1}{z\sqrt{z^2-1}}$
6. $(\csc^{-1} z)' = \frac{-1}{z\sqrt{z^2-1}}$

Proof:-

1) $\cos^{-1} z = \frac{1}{i}\log(z + \sqrt{z^2 - 1})$

$$(\cos^{-1} z)' = \frac{1}{i}\left(\frac{1+\frac{2z}{2\sqrt{z^2-1}}}{z+\sqrt{z^2-1}}\right)$$

$$= \frac{1}{i}\left(\frac{\frac{z+\sqrt{z^2-1}}{\sqrt{z^2-1}}}{z+\sqrt{z^2-1}}\right)$$

$$= \frac{1}{i}\left(\frac{z+\sqrt{z^2-1}}{\sqrt{z^2-1}} \times \frac{1}{z+\sqrt{z^2-1}}\right)$$

$$= \frac{1}{i}\frac{1}{\sqrt{z^2-1}} = \frac{1}{i\sqrt{-(1-z^2)}} = \frac{1}{i^2\sqrt{1-z^2}}$$

$\therefore\ (\cos^{-1} z)' = \frac{-1}{\sqrt{1-z^2}}$

Proof of 2,3,4,5,6 are H.W

Example

find the value of z that satisfied the equation $\cos z = 3$

Solution

$\cos z = 3 \rightarrow z = \cos^{-1} 3$

$$\boldsymbol{\cos^{-1} z = \frac{1}{i}\log\left(z \mp \sqrt{z^2 - 1}\right)}$$

$z = \frac{1}{i}\log\left(3 + \sqrt{9 - 1}\right)$

$z = \frac{1}{i}\log\left(3 + 2\sqrt{2}\right)$

$z = -i\log\left(3 + 2\sqrt{2}\right) + 2\pi k \qquad , k = 0, \mp 1, \dots .$

Example

prove that the solution of the eq. $\cos z = 1$ is $2\pi k \quad , k = 0, \mp 1, \dots$

Solution

$\cos z = 1 \rightarrow z = \cos^{-1} 1$

$$\boldsymbol{\cos^{-1} z = \frac{1}{i}\log\left(z \mp \sqrt{z^2 - 1}\right)}$$

$z = \frac{1}{i}\log\left(1 + \sqrt{1 - 1}\right)$

$z = \frac{1}{i}\log 1 = \frac{1}{i}(0) = 0 + 2\boldsymbol{\pi k} \qquad , \boldsymbol{k = 0, \mp 1}, \dots$

Inverse Trigonometric Hyperbolic Function

1. $sinh^{-1} z = log\left(z + \sqrt{1 + z^2}\right)$
2. $cosh^{-1} z = log\left(z + \sqrt{z^2 - 1}\right)$
3. $tanh^{-1} z = \frac{1}{2} log\left[\frac{1+z}{1-z}\right]$, $z \neq \mp 1$
4. $coht^{-1} z = \frac{1}{2} log\left[\frac{z+1}{z-1}\right]$, $z \neq \mp 1$
5. $sech^{-1} z = log\left[\frac{1+\sqrt{1-z^2}}{z}\right]$
6. $csch^{-1} z = log\left[\frac{1+\sqrt{1+z^2}}{z}\right]$

Proof :-

1) Let $w = \sinh^{-1} z \to z = \sinh w$

$z = \frac{e^{w}-e^{-w}}{2}$ $\qquad\qquad\qquad \times 2e^{w}$

$e^{2w} - 2ze^{w} - 1 = 0$

$e^{w} = \frac{2z \mp \sqrt{4z^2 - 4(1)(-1)}}{2(1)}$

$e^{w} = \frac{2z \mp \sqrt{4z^2 + 4}}{2}$

$e^{w} = \frac{2z \mp 2\sqrt{z^2 + 1}}{2(1)} \to e^{w} = z \pm \sqrt{z^2 + 1}$

$e^{w} = z + \sqrt{z^2 + 1}$

$w = \log\left(z + \sqrt{z^2 + 1}\right)$

$\therefore \sinh^{-1} z = \log\left(z + \sqrt{z^2 + 1}\right)$

Proof 2 → 6 are H.W

Example

Find the value of z such that $\cosh z = \frac{-1}{2}$

Solution

$z = \cosh^{-1}(\frac{-1}{2})$

$cosh^{-1} z = log\left(z + \sqrt{z^2 - 1}\right)$

$\cosh^{-1}(\frac{-1}{2}) = \log\left(\frac{-1}{2} + \sqrt{\frac{1}{4} - 1}\right)$

$= \log\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= \log\left|\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right| + iarg(\frac{-1}{2} + \frac{\sqrt{3}}{2}i)$

$= \log(1) + i(\frac{2\pi}{3} + 2\pi k)$

$= i(\frac{2\pi}{3} + 2\pi k)$

Example

Find the value of $sinh^{-1}(i)$

Solution

$sinh^{-1} z = log\left(z + \sqrt{1 + z^2}\right)$

$sinh^{-1}(i) = log(i + \sqrt{1 - 1})$

$= \log(i)$

$= \log|i| + iarg(i)$

$= \log 1 + i(\frac{\pi}{2} + 2\pi k)$

$= i(\frac{\pi}{2} + 2\pi k)$

Derivative of Inverse Trigonometric Hyperbolic Function

1. $(\sin^{-1} z)' = \frac{1}{\sqrt{1+z^2}}$
2. $(\cosh^{-1} z)' = \frac{-1}{\sqrt{1-z^2}}$
3. $(\tanh^{-1} z)' = \frac{1}{1-z^2} = (\coth^{-1} z)'$
4. $(\text{sech}^{-1} z)' = \frac{-1}{z\sqrt{1-z^2}}$
5. $(\text{csch}^{-1} z)' = \frac{-1}{z\sqrt{1+z^2}}$

Proof

1) $\sinh^{-1} z = \log\left(z + \sqrt{z^2+1}\right)$

$$(\sinh^{-1} z)' = \frac{1+\frac{2z}{2\sqrt{1+z^2}}}{z+\sqrt{1+z^2}}$$

$$= \frac{\sqrt{1+z^2}+z}{\sqrt{1+z^2}} \times \frac{1}{z+\sqrt{1+z^2}}$$

$$\therefore (\sin^{-1} z)' = \frac{1}{\sqrt{1+z^2}}$$

Proof $2 \to 6$ are H.W