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complex analysis

2nd Course

Chapter 5

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Chapter Five

Unit Circle Integral

Integral of the form $(\sin\theta, \cos\theta)$

Let the real integral

$$\int_0^{2\pi} f(\sin\theta, \cos\theta)d\theta$$

to convert the real integral to contour integral (complex) we transform this integral in the unit circle $(r = 1)$

$z = re^{i\theta} \rightarrow \boxed{z = e^{i\theta}} \rightarrow \boxed{dz = ie^{i\theta}d\theta}$

$d\theta = \frac{dz}{ie^{i\theta}} \rightarrow \boxed{d\theta = \frac{dz}{iz}}$

$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \rightarrow \boxed{\sin\theta = \frac{z-z^{-1}}{2i}}$

$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \rightarrow \boxed{\cos\theta = \frac{z+z^{-1}}{2}}$

In general

$\boxed{\sin k\theta = \frac{z^k-z^{-k}}{2i}}$ and $\boxed{\cos k\theta = \frac{z^k+z^{-k}}{2}}$

$$\int_0^{2\pi} f(\sin\theta, \cos\theta)d\theta = \int_c f\left(\frac{z-z^{-1}}{2i}, \frac{z+z^{-1}}{2}\right)\frac{dz}{iz}$$

this integral can be evaluated by Cauchy integral theorem and the Residues theorem

Example

Convert this integral to contour integral and evaluate it .

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$$

Solution

Let $z = e^{i\theta} \rightarrow d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{z+z^{-1}}{2}$

$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_c \frac{1}{2+\frac{z+z^{-1}}{2}} \frac{dz}{iz}$

$\int_c \frac{2}{4+z+z^{-1}} \frac{dz}{iz} = \frac{2}{i} \int_c \frac{dz}{z^2+4z+1}$

The singular Points are : $z = \frac{-4\mp\sqrt{16-4(1)(1)}}{2(1)} \begin{cases} \rightarrow & -2+\sqrt{3} \in C \\ \rightarrow & -2-\sqrt{3} \notin C \end{cases}$

$Res[f, -2+\sqrt{3}] = \lim_{z\to -2+\sqrt{3}} (z+2-\sqrt{3}) \frac{2}{i} \frac{1}{(z+2-\sqrt{3})(z+2+\sqrt{3})}$

$= \frac{2}{i} \frac{1}{-2+\sqrt{3}+2+\sqrt{3}} = \frac{1}{\sqrt{3}i}$

$\therefore \int_c f(z)dz = 2\pi i \, Res[f, -2+\sqrt{3}]$

$= 2\pi i \frac{1}{\sqrt{3}i} = \frac{2}{\sqrt{3}}\pi$

Example

Convert this integral to contour integral and evaluate it .

$$\int_0^{2\pi} \frac{1}{5+4\sin\theta} d\theta$$

H.W

Example

show that $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta = \frac{25}{12}\pi$

Solution

Let $z = e^{i\theta} \rightarrow \quad d\theta = \frac{dz}{iz}$

$\cos\theta = \frac{z+z^{-1}}{2} \rightarrow \quad \cos 3\theta = \frac{z^3+z^{-3}}{2}$

$$\therefore \int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta = \int_c \frac{\left(\frac{z^3+z^{-3}}{2}\right)}{5-4\left(\frac{z+z^{-1}}{2}\right)} \frac{dz}{iz}$$

$$\frac{1}{2i}\int_c \frac{z^3+\frac{1}{z^3}}{5z-2z^2-2}\, dz = \frac{1}{2i}\int_c \frac{z^6+1}{-2z^5+5z^4-2z^3}$$

$$\frac{-1}{2i}\int_c \frac{z^6+1}{z^3(2z^2-5z+2)} = \frac{-1}{2i}\int_c \frac{z^6+1}{z^3(2z-1)(z-2)} dz$$

sing. Point $z = 0 \in C$; $z = \frac{1}{2} \in C$; $z = 2 \notin C$

to calculate $Res[f, 0]$ repeated 3-times

$$\emptyset(z) = \lim_{z\to 0}(z-0)^3 \frac{z^6+1}{z^3(2z^2-5z+2)} = \frac{1}{2} \neq 0$$

$$Res[f, 0] = \frac{1}{2!}\lim_{z\to 0} \frac{d^2}{dz^2} \frac{z^6+1}{2z^2-5z+2} = \frac{5}{8}$$

$$Res\left[f, \frac{1}{2}\right] = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right) \frac{z^6+1}{z^3(2z-1)(z-2)} = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right) \frac{z^6+1}{2z^3\left(z-\frac{1}{2}\right)(z-2)}$$

$$= \frac{65/64}{(-3/2)(1/4)} = \frac{-65}{24}$$

$$\therefore \int_c = \frac{-1}{2i}\left(2\pi i\left[\frac{5}{8} - \frac{65}{24}\right]\right) = \frac{25}{12}\pi$$

Example

Convert this integral to contour integral and evaluate it .

$$\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta$$

H.W

Example

Convert this integral to contour integral and evaluate it .

$$\int_0^{2\pi} \frac{1}{2 + \sin\theta} d\theta$$

Solution

$$\int_0^{2\pi} \frac{1}{2+\sin\theta} d\theta = \int_c \frac{1}{2+\frac{z-z^{-1}}{2i}} \frac{dz}{iz}$$

$$\int_c \frac{1}{2iz+\frac{z^2-1}{2}} dz = \int_c \frac{2}{z^2+4iz-1} dz$$

$$\text{Sing. P.} \begin{cases} \rightarrow \quad (-2-\sqrt{3})i \notin C \\ \rightarrow \quad (-2+\sqrt{3})i \in C \end{cases}$$