Ministry of Higher Education and Scientific Research

University of Mosul

College of Computer Science and Mathematics

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6

COMLEX ANALYSIS

1st Course

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CHAPTER SIX

INTEGRAL IN COMPLEX PLANE

Euler Formula :

if θ is the angle (Argument) of the components of the complex function then Euler formula can written as :

$$e^{i\theta} = cos\theta + isin\theta \quad , \quad 0 \le \theta \le 2\pi$$

$$\overline{e^{i\theta}} = cos\theta - isin\theta \quad , \quad 0 \le \theta \le 2\pi$$

Note : $z = x + yi = rcos\theta + irsin\theta = r(cos\theta + isin\theta) = re^{i\theta}$

$|z| = |re^{i\theta}| = r \ , such\ that\ |e^{i\theta}| = 1$

The graph of function $|z| = r$ is a circle with center is the origin and radius is r , while the graph of function $|z - z_o| = r$ is a circle with center is the $z_0 = (x_0, y_0)$ and radius is r .

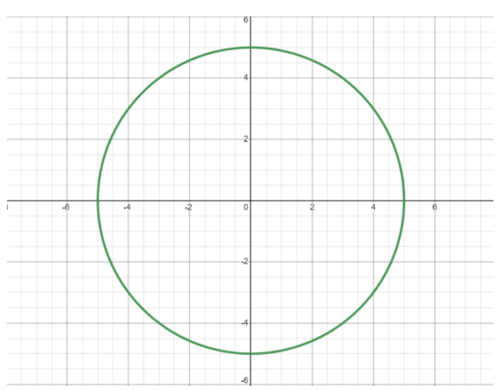
Example

Sketch the following graph in complex plane

Solution

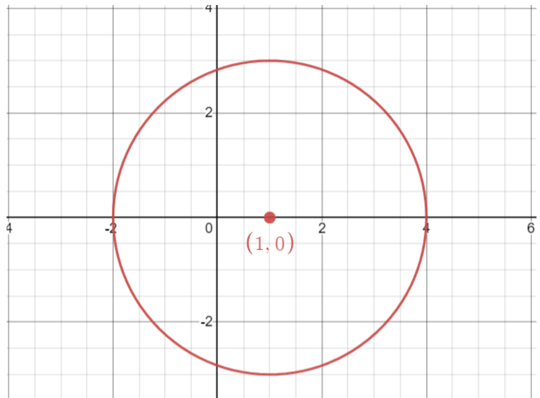
1. $|z| = 5$

represent a circle with center is origin and radius is 5



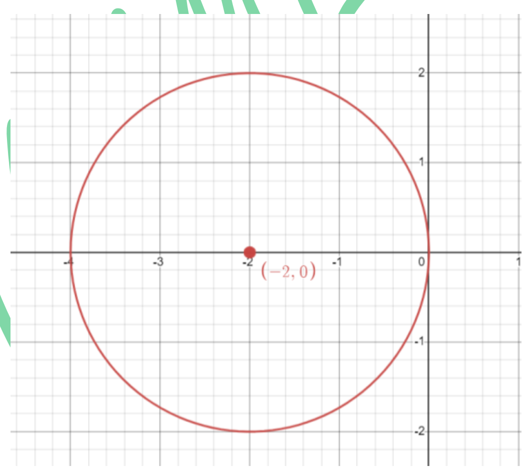
2. $|z-1| = 3$

represent a circle with center is (1,0) and radius is 3



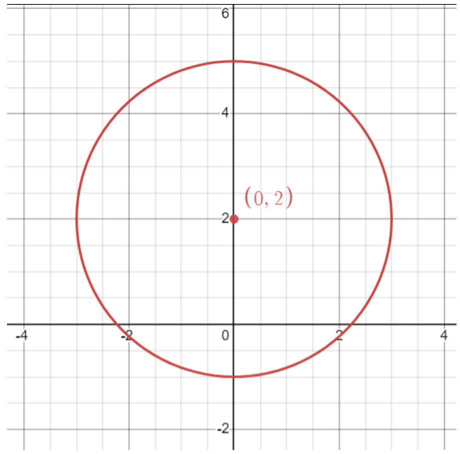
3. $|z+2| = 2$

represent a circle with center is (-2,0) and radius is 2



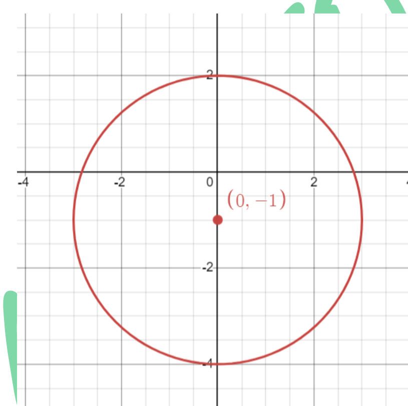
4. $|z-2i| = 3$

represent a circle with center is (0,2) and radius is 3



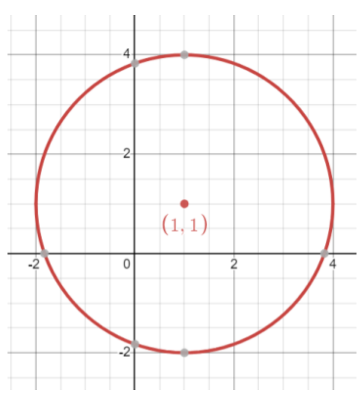
5. $|z + i| = 3$

represent a circle with center is (0,-1) and radius is 3



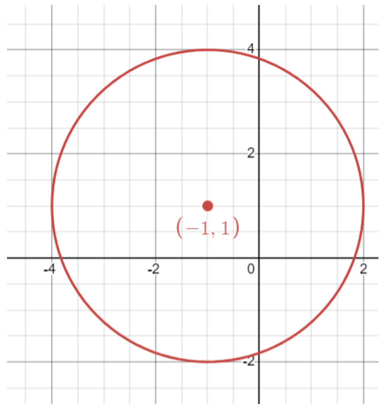
6. $|z - 1 - i| = 3$

represent a circle with center is (1,1) and radius is 3



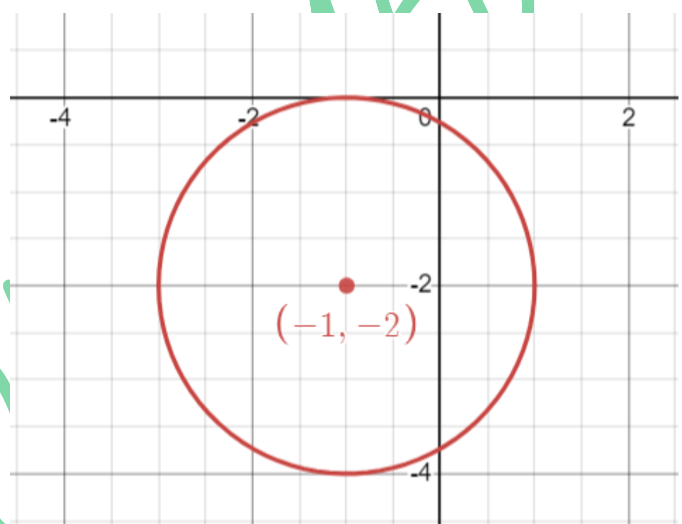
7. $|z + 1 - i| = 3$

represent a circle with center is (-1,1) and radius is 3



8. $|z + 1 + 2i| = 2$

represent a circle with center is (-1,-2) and radius is 2



Integral in Complex Plane

We have introduced functions of a complex variable. We also established when functions are differentiable as complex functions, or holomorphic. In this chapter we will turn to integration in the complex plane. We will learn how to compute complex path integrals, or contour integrals. We will see that contour integral methods are also useful in the computation of some of the real integrals that we will face when exploring the residues and poles .

the types of integration are : Direct integral , Paths integral , counter integral and cauchy's integrals

Direct Integral :

Let $f(t) = u(t) + iv(t)$ where u and v be real valued function , the definite integral of $f(t)$ on the interval $a \le t \le b$ is defined as :

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt$$

Example 1

Evaluate $\int_0^1 (1 + it)^2 dt$

Solution

$$I = \int_0^1 (1 + 2it - t^2)dt$$

$$I = \int_0^1 (1 - t^2)dt + i \int_0^1 2tdt$$

$$I = t - \frac{t^3}{3}\Big|_0^1 + i\, t^2|_0^1 = \frac{2}{3} + i$$

Example 2

Evaluate $\int_0^{\frac{\pi}{4}} e^{it}\, dt$

Solution

$$I = \int_0^{\frac{\pi}{4}} e^{it}\, dt = \int_0^{\frac{\pi}{4}} cost dt + i \int_0^{\frac{\pi}{4}} sint\ \ dt$$

$$I = sint|_0^{\frac{\pi}{4}} - i\, cost|_0^{\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + i(1 - \frac{1}{\sqrt{2}})$$

Example 3

1) Evaluate the following integral :

A) $\int_1^2 \left(\frac{1}{t} - i\right)^2 dt$

$B)\ \int_0^{\frac{\pi}{6}} e^{2it} dt$

$C) \int_0^{\pi} e^{(1+i)t}\, dt$

H.W

Integral on Regions

Example 4

if C is the region defined by a circle $z - z_o = re^{i\theta}$ such that $0 \leq \theta \leq 2\pi$, and z_o is the center of the circle , r is the radius , find

$$\int_C \frac{dz}{z - z_o}$$

Solution

$$z - z_o = re^{i\theta} \quad \rightarrow dz = ire^{i\theta} d\theta$$

$$\int_C \frac{dz}{z - z_o} = \int_0^{2\pi} \frac{ire^{i\theta} d\theta}{re^{i\theta}} = \int_0^{2\pi} i\, d\theta = i\theta|_0^{2\pi} = 2\pi i$$

Example 5

if C is the region defined by right half of a circle $|z| = 1$, find $\int_C |z|\, dz$

Solution

$$|z| = 1 , z = e^{i\theta} \quad \rightarrow dz = ie^{i\theta} d\theta$$

$$\int_C |z| dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1.ie^{i\theta} d\theta = e^{i\theta}\Big|_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$$

$$\int_C |z| dz = e^{i\frac{\pi}{2}} - e^{-i\frac{\pi}{2}} = 2i$$

(check another method by Euler formula expansion integral)

Example 6

if C is the portion (section) of circle defined by 1st and 3rd Quarter of circle $|z| = 2$, find $\int_C |z|\, dz$

Solution

The region C divided into two sub-regions C:C_1+C_2

$|z| = 2\,, z = 2e^{i\theta} \;\; \rightarrow dz = 2ie^{i\theta}d\theta$

$$\int_C |z|dz = \int_{C_1} |z|\,dz + \int_{C_2} |z|\,dz$$

$$\int_{C_1} |z|dz = \int_0^{\frac{\pi}{2}} 2.2ie^{i\theta}d\theta = 4e^{i\theta}\Big|_0^{\frac{\pi}{2}} = 4(i-1)$$

$$\int_{C_2} |z|dz = \int_\pi^{\frac{3\pi}{2}} 2.2ie^{i\theta}d\theta = 4e^{i\theta}\Big|_\pi^{\frac{3\pi}{2}} = 4(1-i)$$

$$\int_C |z|dz = 4i - 4 + 4 - 4i = 0$$

Example 7

if C is the upper half of circle $|z| = 2$, find $\int_C z^2\, dz$

Solution :

$|z| = 2\,, z = 2e^{i\theta} \;\; \rightarrow dz = 2ie^{i\theta}d\theta$

$$\int_C z^2 dz = \int_0^\pi z^2\, dz$$

$$\int_C z^2 dz = \int_0^{\pi} (2e^{i\theta})^2 . 2ie^{i\theta} d\theta$$

$$\int_C z^2 dz = 8\int_0^{\pi} ie^{3i\theta} d\theta = \frac{8}{3} e^{3i\theta}\Big|_0^{\pi} = \frac{8}{3}\left(e^{3\pi i} - 1\right) = \frac{-16}{3}$$

Example 8

if C is the lower half of circle $|z| = 1$, find $\int_C \frac{1}{\sqrt{z}} dz$

Solution

$$|z| = 1 , z = e^{i\theta} \rightarrow dz = ie^{i\theta} d\theta$$

$$\int_C \frac{1}{\sqrt{z}} dz = \int_{\pi}^{2\pi} \frac{ie^{i\theta} d\theta}{e^{i\theta/2}} = \int_{\pi}^{2\pi} ie^{i\frac{\theta}{2}} d\theta$$

$$\int_C \frac{1}{\sqrt{z}} dz = 2e^{i\frac{\theta}{2}}\Big|_{\pi}^{2\pi} = 2(i - 1)$$

Example 9

if C is the upper half of circle $|z| = 1$, find $\int_C z^n dz$, $n \in Z^+$

Solution

$$|z| = 1 , z = e^{i\theta} \rightarrow dz = ie^{i\theta} d\theta$$

$$\int_C z^n dz = \int_0^{\pi} z^n dz$$

$$\int_C z^n dz = \int_0^{\pi} \left(e^{i\theta}\right)^n . ie^{i\theta} d\theta$$

$$\int_C z^n dz == \int_0^{\pi} i\, e^{i(n+1)\theta} d\theta = \left.\frac{e^{i(n+1)\theta}}{n+1}\right|_0^{\pi} = \frac{1}{n+1}\left((-1)^n - 1\right)$$

Example 10

if C is the left half of circle $|z| = 64$, find $\int_C \frac{\sqrt[3]{z}-2}{\sqrt{z}} dz$

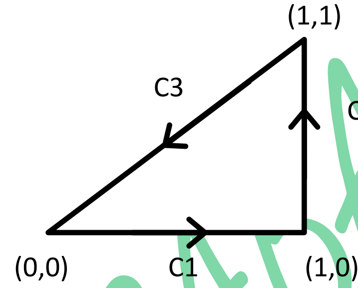
H.W

Integral at paths

1) If the given region represent straight line (horizontal) then the value of y stay constant $(dy = 0)$ and x is changing
2) If the given region represent straight line (vertical) then the value of x stay constant $(dx = 0)$ and y is changing
3) If the given region represent straight line (italic) then x and y changing and we will find a relation connecting x with y by slope and point rule .

Example 11

Find the numerical value to $\int_C xdz$, if C:$C_1+C_2+C_3$ defined in the given figure :



Solution

$\boldsymbol{C1}: (\mathbf{0},\mathbf{0}) \rightarrow (\mathbf{1},\mathbf{0})$

$y = 0 \rightarrow dy = 0 \ , \qquad x: 0 \rightarrow 1$

$$\int_{C1} xdz = \int_{C1} x(dx + idy)$$

$$\int_{C1} xdz = \int_0^1 xdx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}$$

$\boldsymbol{C2}: (\mathbf{1},\mathbf{0}) \rightarrow (\mathbf{1},\mathbf{1})$

$x = 1 \rightarrow dx = 0 \ , \qquad y: 0 \rightarrow 1$

$$\int_{C2} xdz = \int_{C2} x(dx + idy) = \int_0^1 idy$$

$$\int_{C2} xdz = iy|_0^1 = i$$

$\boldsymbol{C3}: (\mathbf{1},\mathbf{1}) \rightarrow (\mathbf{0},\mathbf{0})$

$$\frac{y - y1}{x - x1} = m \ , \qquad such\ that\ \ m = \frac{\Delta y}{\Delta x}$$

$$\frac{y-0}{x-0}=\frac{0-1}{0-1} \rightarrow y=x \rightarrow dy=dx$$

$$\int_{C3} xdz=\int_{C3} x(dx+idy)$$

$$\int_{C3} xdz=\int_{1}^{0} x(dx+idx)=(1+i)\int_{1}^{0} x\,dx$$

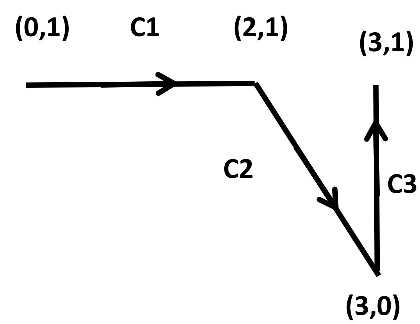
$$\int_{C3} xdz=(1+i)\frac{x^2}{2}\Big|_{1}^{0}=\frac{-1}{2}(1+i)$$

$$\therefore \int_{C} xdz=\int_{C1} xdz+\int_{C2} xdz+\int_{C3} xdz$$

$$\therefore \int_{C} xdz=\frac{1}{2}+i-\frac{1}{2}(1+i)=\frac{i}{2}$$

Example 12

Find the numerical value to $\int_C \bar{z}dz$, if C:C_1+C_2+C_3 defined in the given figure :



Solution

$z=x+yi \rightarrow \bar{z}=x-yi\,,dz=dx+idy$

$C1: (0,1) \rightarrow (2,1)$

$y=1 \rightarrow dy=0\ ,\qquad x:0\rightarrow 2$

$$\int_{C1} \bar{z}dz = \int_{C1} (x - yi)(dx + idy)$$

$$\int_{C1} \bar{z}dz = \int_0^2 (x - i)dx = \frac{(x - i)^2}{2}\bigg|_0^2 = \frac{(2 - i)^2}{2} + \frac{1}{2} = 2 - 2i$$

$C2: (2, 1) \rightarrow (3, 0)$

$$\frac{y - 1}{x - 2} = \frac{0 - 1}{3 - 2} \rightarrow y - 1 = 2 - x \rightarrow y = 3 - x \rightarrow dy = -dx$$

$$\int_{C2} \bar{z}dz = \int_{C2} (x - yi)(dx + idy)$$

$$\int_{C2} \bar{z}dz = \int_{C2} (x - (3 - x)i)(dx - idx)$$

$$\int_{C2} \bar{z}dz = (1 - i)\int_2^3 (x - 3i + xi)dx$$

$$\int_{C2} \bar{z}dz = (1 - i)\left\{\frac{x^2}{2} - 3ix + i\frac{x^2}{2}\bigg|_2^3\right\} = 2 - 3i$$

$C3: (3, 0) \rightarrow (3, 1)$

$x = 3 \rightarrow dx = 0$, $\quad y: 0 \rightarrow 1$

$$\int_{C3} \bar{z}dz = \int_0^1 (x - yi)(dx + idy) = \int_0^1 (3 - yi)idy$$

$$\int_{C3} \bar{z}dz = \int_0^1 (3i + y)dy$$

$$\int_{C3} \bar{z}dz = 3iy + \frac{y^2}{2}\Big|_0^1 = 3i + \frac{1}{2}$$

$$\therefore \int_{C} \bar{z}dz = \int_{C1} \bar{z}dz + \int_{C2} \bar{z}dz + \int_{C3} \bar{z}dz = 2 - 2i + 2 - 3i + 3i + \frac{1}{2} = \frac{5}{2} - 2i$$

Example 13

Draw the paths and Find the numerical value to $\int_C (2z^2 - 3\bar{z} + 1)dz$, if $C: C_1 + C_2 + C_3$ defined in the given figure :

$\boldsymbol{C1 : (1,2) \to (1,5)}$, $\boldsymbol{C2 : (1,5) \to (3,5)}$, $\boldsymbol{C3 : (3,5) \to (7,2)}$

H.W