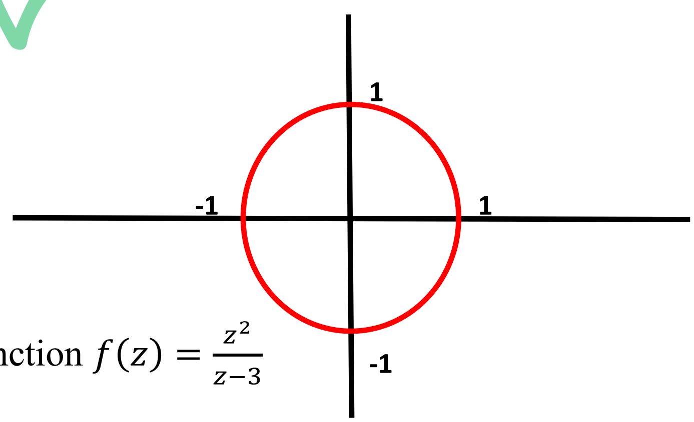
Cauchy's Corsat Theorem in Integral :

The function that continuous with continuous derivative is called analytic function , if the function $f(z) = u + iv$ is analytic on the region C and continuous , then the integral :

$$\oint_C f(z)dz = 0$$

Notes

- In Complex region the problems in functions comes at most from fractional functions , but the radical function in complex has no any problem .
- The singular point is the point that makes the function not continuous i.e the value of function is undefined



Example 14

Prove that $\oint_C \frac{z^2}{z-3} dz = 0$ **at** $|z| = 1$

Solution

By applying cauchy's Theorem in integral the function $f(z) = \frac{z^2}{z-3}$

The singular point is $3 \notin C$, and the function analytic on all values except the value (z=3) but z=3 don't lies in circle $|z| = 1$ or on its boundary , then by cauchy's theorem in integral is :

$$\oint_C \frac{z^2}{z-3} dz = 0$$

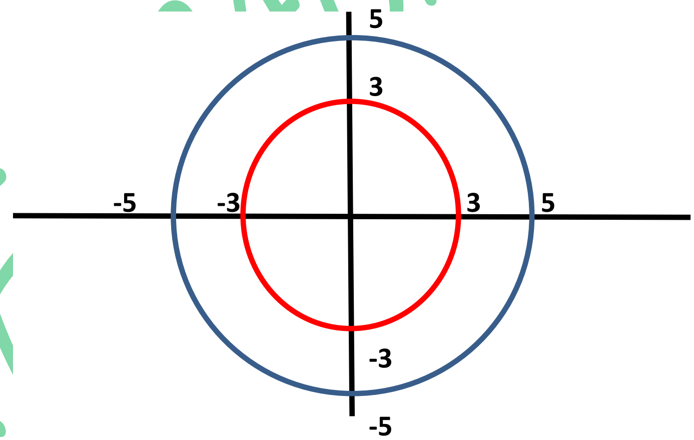
Example 15

Find the Integral

$$\oint_C \frac{1}{(z-1)^2(z^2+4)}\,dz$$

Such that C is ring between $C_1: |z| = 3$, $C_2: |z| = 5$

Solution



By applying cauchy's Theorem in

integral the function $f(z) = \frac{1}{(z-1)^2(z^2+4)}$

Is analytic on the ring $3 \le |z| \le 5$,

Because the singular points for the function f(z) is $(1\ , 2i\ , -2i\ ) \notin C$, and this point don't lies in the ring : C

then by cauchy's theorem in integral is :

$$\oint_C \frac{1}{(z-1)^2(z^2+4)}\,dz = 0$$

Example 16

Find all the integrals at the region C

A)

$$\oint_C \frac{z-4}{z^2-6z+5}dz \quad , \qquad C:|z|=\frac{1}{2}$$

B)

$$\oint_C \frac{z}{z^2+6iz-8}dz \quad , \qquad C:|z|=1$$

H.W

First Cauchy's Integral Formula (C.I.F.1) :

Let C be a region and f is analytic function in all values inside C except $(z = a)$ such that $z = a$ is a singular point then :

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - a} dz$$

i.e

$$\oint_C \frac{f(z)}{z - a} dz = 2\pi i\, f(a)$$

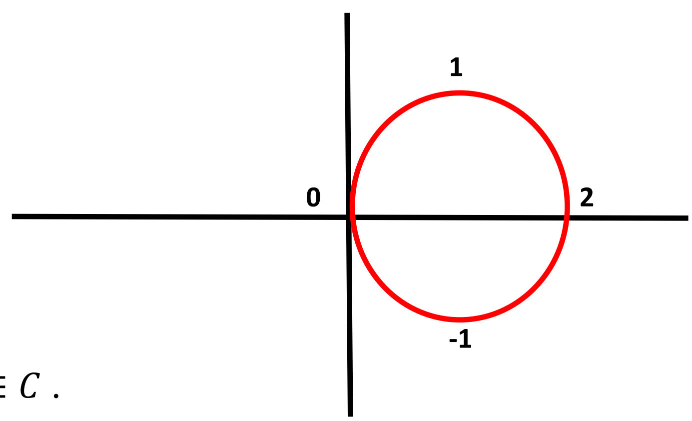
Example 17

Find the Integral on the following regions :

$$\oint_C \frac{z^2 + 1}{(z^2 - 1)} dz$$

Such that:

A) $C: |z - 1| = 1$



Solution

The singular point is $z = -1 \notin C$, $z = 1 \in C$.

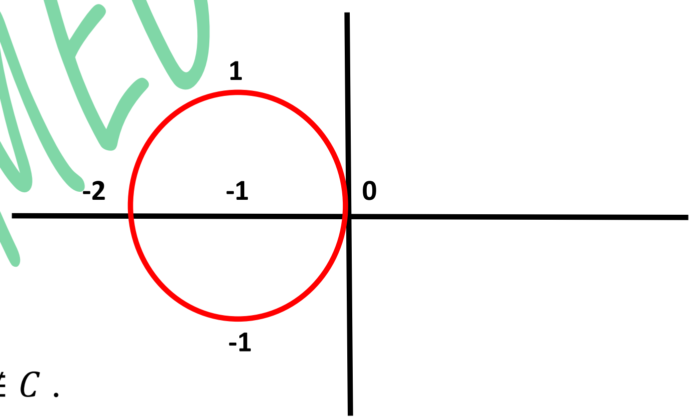
Then the function is analytic on all points in C except $z = 1$

$$\oint_C \frac{z^2+1}{z^2-1}dz = \oint_C \frac{z^2+1}{(z-1)(z+1)}dz$$

$$\oint_C \frac{z^2+1}{z^2-1}dz = \oint_C \underbrace{\frac{z^2+1}{(z+1)}}_{\substack{f(z)\ new \\ analytic}} \frac{1}{(z-1)}dz$$

$$\oint_C \frac{z^2+1}{z^2-1}dz = 2\pi i\, f(1) = 2\pi i\left(\frac{1^2+1}{1+1}\right) = 2\pi i$$

B) $C: |z+1| = 1$



Solution

The singular point is $z = -1 \in C$, $z = 1 \notin C$.

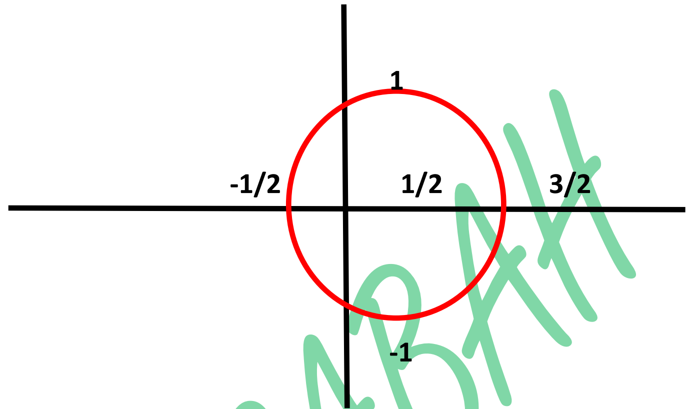
Then the function is analytic on all points in C except $z = -1$

$$\oint_C \frac{z^2+1}{z^2-1}dz = \oint_C \frac{z^2+1}{(z-1)(z+1)}dz$$

$$\oint_C \frac{z^2+1}{z^2-1}dz = \oint_C \frac{z^2+1}{(z-1)} \frac{1}{(z+1)}dz$$

$$\oint_C \frac{z^2+1}{z^2-1}dz = 2\pi i\, f(-1) = 2\pi i\left(\frac{(-1)^2+1}{-1-1}\right) = -2\pi i$$

C) $C: \left|z - \frac{1}{2}\right| = 1$



Solution

The singular point is $z = 1 \in C$, $z = -1 \notin C$.

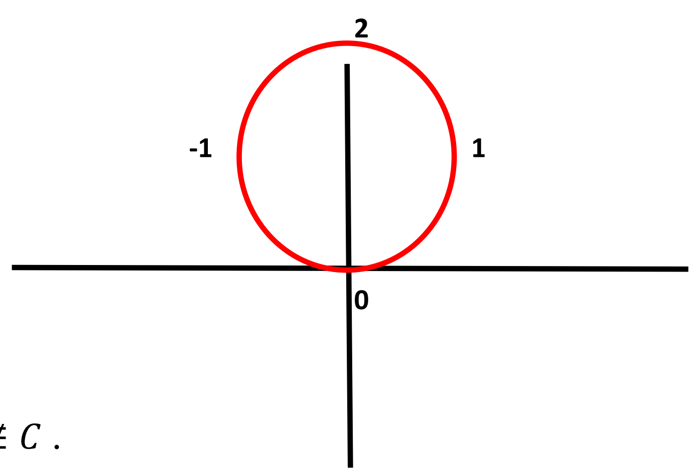
Then the function is analytic on all points in C except $z = 1$

$$\oint_C \frac{z^2+1}{z^2-1}dz = \oint_C \frac{z^2+1}{(z-1)(z+1)}dz$$

$$\oint_C \frac{z^2+1}{z^2-1}dz = \oint_C \frac{z^2+1}{(z+1)}\frac{1}{(z-1)}dz$$

$$\oint_C \frac{z^2+1}{z^2-1}dz = 2\pi i\, f(1) = 2\pi i\left(\frac{(1)^2+1}{1+1}\right) = 2\pi i$$

D) $C: |z - i| = 1$



Solution

The singular point is $z = -1 \notin C$, $z = 1 \notin C$.

Then the function is analytic on all points in C

Then by cauchy's - corsat theorem

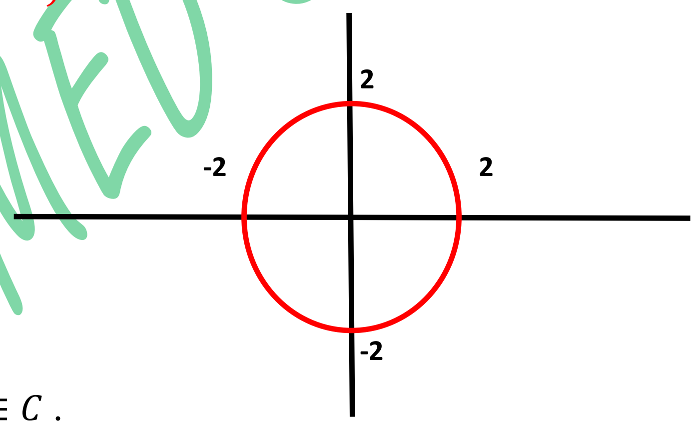
$$\oint_C \frac{z^2+1}{z^2-1}dz = 0$$

Example 18

Find the Integral on the region C :

$$\oint_C \frac{1}{(z^2-1)}dz$$

Such that $C: |z| = 2$



Solution

The singular point is $z = -1 \in C$, $z = 1 \in C$.

Then the function is analytic on all points in C except z=1 , z=-1

Since the singular point are two points then the integral will be divided into two regions C_1 and C_2

let $z = 1 \in C_1$, $z = -1 \in C_2$

$$\oint_C \frac{1}{z^2-1}dz = \oint_C \frac{1}{(z-1)(z+1)}dz$$

$$\oint_C \frac{1}{z^2-1}dz = \oint_{C_1} \frac{dz}{(z+1)}\frac{1}{(z-1)} + \oint_{C_2} \frac{dz}{(z-1)}\frac{1}{(z+1)}$$

$$\oint_C \frac{1}{z^2-1}dz = 2\pi i\, f(1) + 2\pi i\, f(-1) = 2\pi i\,\frac{1}{2} + 2\pi i\,\frac{-1}{2} = 0$$

Notes :

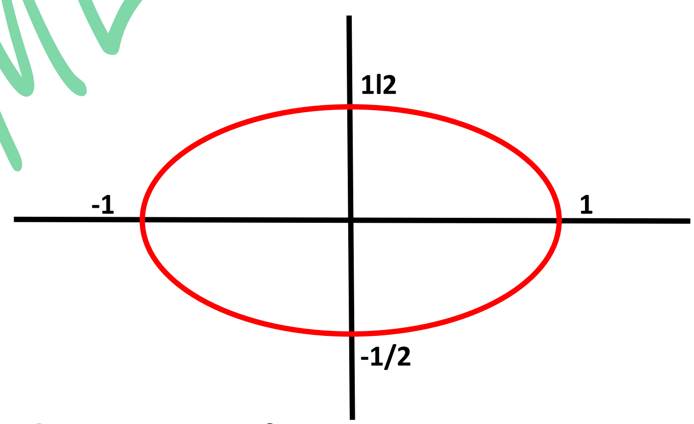
- if the function is analytic then the value of the integral is zero.
- if the integral value is zero then not necessary to be the function is analytic

Example 19

Find the Integral on the region C :

$$\oint_C \frac{sinz}{z}dz$$

Such that: $C: x^2 + 4y^2 = 1$



Solution

The singular point is $z = 0 \in C$.

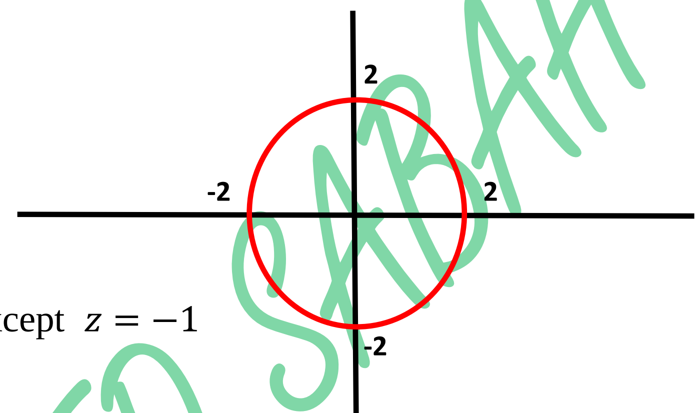
Then the function is analytic on all points in C except $z = 0$

$$\oint_C \frac{sinz}{z}dz = \oint_C sinz\,\frac{dz}{z} = 2\pi i\, f(0) = 2\pi i\, sin(0) = 0$$

Example 20

Find the Integral on the region C :

$$\oint_C \frac{e^z}{z+1} dz$$



Such that: $C: |z| = 2$

Solution

The singular point is $z = -1 \in C$.

the function is analytic on allpoints in C except $z = -1$

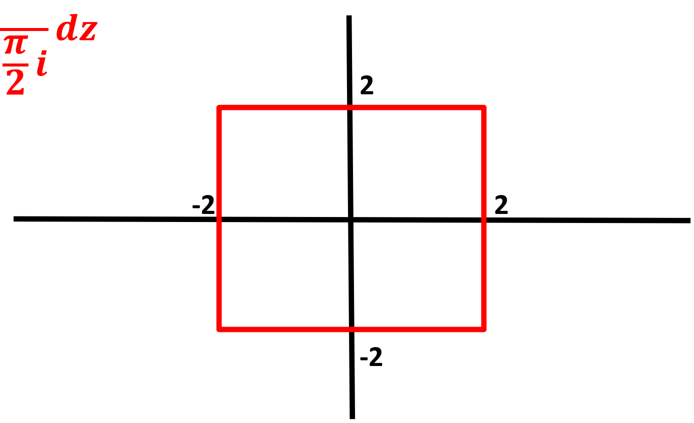
$$\oint_C \frac{e^z}{z+1} dz = \oint_C e^z \frac{1}{(z+1)} dz$$

$$\oint_C \frac{e^z}{z+1} dz = 2\pi i\, f(-1) = 2\pi i\, e^{-1}$$

Example 21

Find the Integral on the region C that bounded by the square determinate by the lines $x = \pm 2$, $y = \pm 2$:

$$\oint_C \frac{e^{-z}}{z - \frac{\pi}{2} i} dz$$



Solution

The singular point is $z = \frac{\pi}{2} \in C$ (note that $\pi = 3.14$.)

Then the function is analytic on all points in C except $z = \frac{\pi}{2} \in C$

$$\oint_C \frac{e^{-z}}{z - \frac{\pi}{2} i} dz = \oint_C e^{-z} \frac{1}{(z - \frac{\pi}{2} i)} dz$$

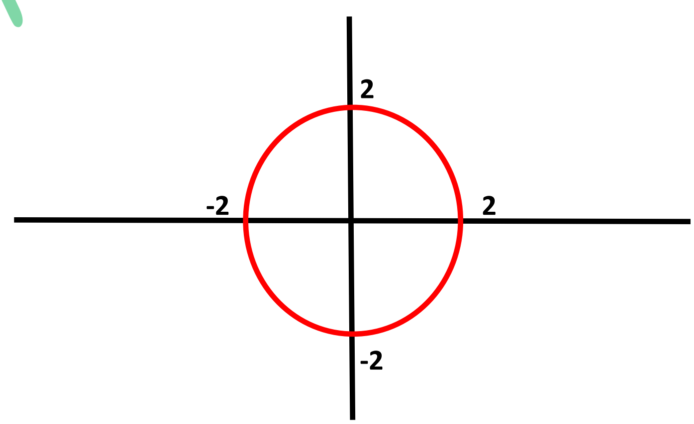
$$\oint_C \frac{e^{-z}}{z - \frac{\pi}{2} i} dz = 2\pi i \, f\left(\frac{\pi}{2} i\right) = 2\pi i \, (-i) = 2\pi$$

Example 22

Find the Integral on the region C :

$$\oint_C \frac{e^{\pi z}}{(z-1)(z-i)} dz$$

Such that: $C: |z| = 2$



Solution

The singular point is $z = 1, i \in C$.

Then the function is analytic on all points in C except $z = 1, i$

Since the singular point are two points then the integral will be divided into two regions C_1 and C_2

$$\oint_C \frac{e^{\pi z}}{(z-1)(z-i)}dz = \oint_{C_1} \frac{e^{\pi z}}{z-i}\frac{1}{(z-1)}dz + \oint_{C_2} \frac{e^{\pi z}}{z-1}\frac{1}{(z-i)}dz$$

$$\oint_C \frac{e^{\pi z}}{(z-1)(z-i)}dz = 2\pi i\, f(1) + 2\pi i f(i) = \frac{2\pi i(e^{\pi}+1)}{1-i}$$

Example 23

A) Evaluate the integral in the following regions

$$\oint_C \frac{z}{z^2+1}dz$$

Such that C is:

1) $C: |z-i| = 1$, 2) $C: |z+i| = 1$

3) $C: |z| = 2$, 4) $C: |z-i| = 3$

B) Evaluate the integral in the following regions

$$\oint_C \frac{z}{(9-z^2)(z+i)}dz \ , \quad C: |z| = 2$$

C) Evaluate the integral in the following regions

$$\oint_C \frac{z^2-z+1}{z^2-3z+2}dz$$

Such that C is:

1) $C: |z+1| = 2$

2) $C: |z+1| = 3$

3) $C: ring\ ,\quad 1 \le |z| \le 3$

Solution : H.W

Second Cauchy's Integral Formula (C.I.F.2) :

Let C be a region and f is analytic function with analytic derivative in all values inside C except (z=a) such that z=a is a singular point then :

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz$$

i.e

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!}\, f^{(n)}(a)$$

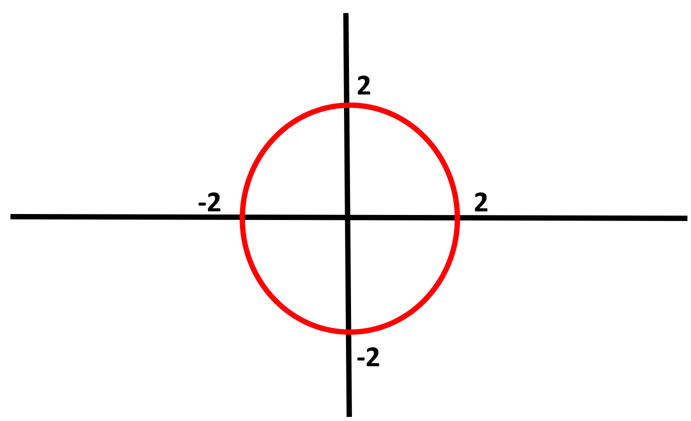
Note :

C.I.F.2 used when the singular points are repeated

Example 24

Find the Integral on the following regions :

$$\oint_C \frac{\cosh(z)}{z^4}dz \quad , \quad C: |z| = 2$$



Solution

The singular point is $z = 0 \in C$ $\;(repeated\ 4\ times\,)$.

Then the function is analytic on all points in C except $z = 0$

$$\oint_C \frac{coshz}{z^4}dz = \oint_C \frac{coshz}{(z-0)^4}dz$$

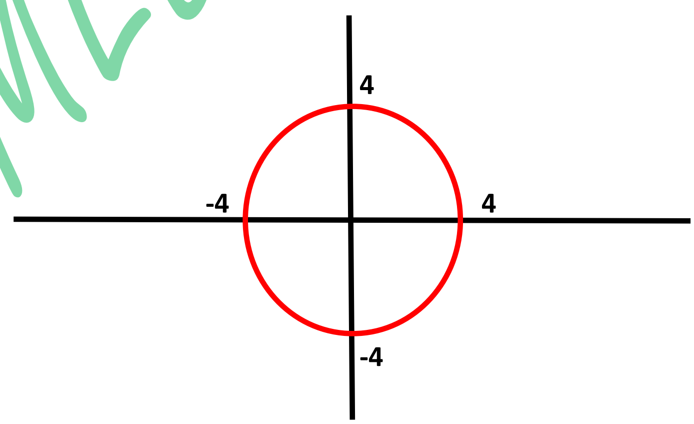
n+1=4 , n=3

$$\oint_C \frac{coshz}{z^4}dz = \frac{2\pi i}{3!}f'''(0) = \frac{2\pi i}{3!}(sinh0) = 0$$

Example 25

Find the Integral on the following regions :

$$\oint_C \frac{1}{(z^2+4)^2}dz \quad , \quad C: |z| = 4$$



Solution

The singular point is $z = \pm 2i \in C$ $(\,repeated\ 2\ times\,)$.

Then the function is analytic on all points in C except z=2i , -2i

$$\oint_C \frac{1}{(z^2+4)^2}dz = \oint_{C_1} \frac{1}{(z+2i)^2}\frac{dz}{(z-2i)^2} + \oint_{C_2} \frac{1}{(z-2i)^2}\frac{dz}{(z+2i)^2}$$

n+1=2 , n=1

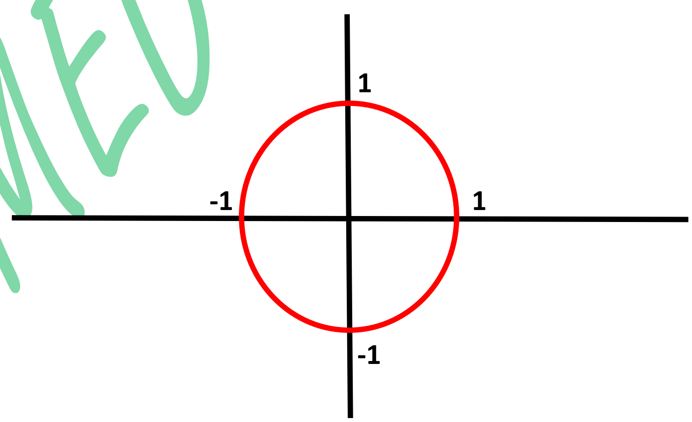
$$\oint_C \frac{1}{(z^2+4)^2}dz = \frac{2\pi i}{1!}f'(2i) + \frac{2\pi i}{1!}f'(-2i)$$

$$\oint_C \frac{1}{(z^2+4)^2}dz = 2\pi i\frac{-2}{(4i)^3} + 2\pi i\frac{-2}{(-4i)^3} = 0$$

Example 26

Find the Integral on the following regions :

$$\oint_C z^{-2}\, e^{-z} sinz\, dz \quad , \quad C: |z| = 1$$



Solution

The singular point is $z = 0 \in C$ $\left(repeated\ 2\ times \right)$.

Then the function is analytic on all points in C except $z = 0$

$$\oint_C \frac{\sin z}{z^2 e^z}dz = \oint_C \frac{\sin z}{e^z}\frac{dz}{z^2}$$

n+1=2 , n=1

$$\oint_C \frac{\sin z}{z^2 e^z} dz = \frac{2\pi i}{1!} f'(0) = 2\pi i(1) = 2\pi i$$

Example 27

Find the Integral on the following regions :

$$\oint_C \frac{2z-3}{z^4 - 8z^2 + 16} dz$$

in the following regions :

A) $C: |z| = 1$
B) $C: |z| = 3$
C) $C: |z - 2| = 2$
D) $C: |z + 1| = 1$

H.W