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complex analysis

2nd Course

Chapter 6

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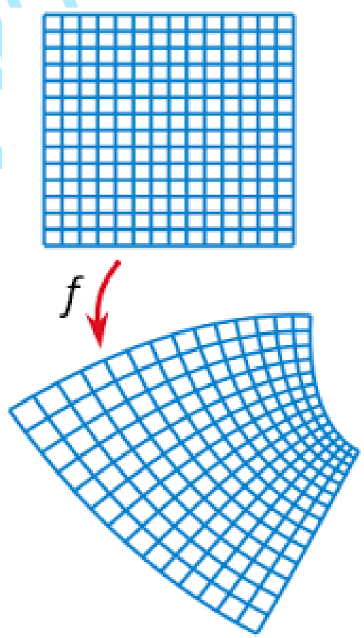
Chapter Six

The Conformal Mapping

Conformal mapping is a fundamental concept in complex analysis that plays a crucial role in understanding the behavior and properties of complex functions. It involves transforming one complex plane onto another while preserving the angles between intersecting curves. In simpler terms, conformal mappings are transformations that maintain the local shape of figures.

in math **Conformal mapping** ia a transformation of one graph into another graph in such away that the angle of intersection of any two lines or curves remains unchanged , the most common example is the Mercator map , a two dimensional representation of the surface of the earth that preserves compass direction other compass direction other conformal maps , some times called orthomorphic projections , preserves angle but not shapes .



Conformal mappings also find extensive applications in fluid dynamics, electromagnetism, and quantum mechanics. In fluid dynamics, conformal mappings are used to model complex flows around obstacles and analyze the behavior of fluids in different geometries. In electromagnetism, conformal mappings help solve boundary value problems and study the behavior of electric and magnetic fields in various configurations. In quantum mechanics, conformal mappings are utilized to transform problems in complex geometries into simpler ones that are easier to analyze.

Conformal mapping encompasses various types of transformations that preserve angles locally while mapping one complex plane onto another. These transformations play a crucial role in complex analysis, geometry, physics, and engineering. Here are some common types of conformal mappings :

Linear Transformations: These are the simplest conformal mappings, defined by functions of the form $f(z) = az + b$, where a and b are complex constants. Linear transformations include translations, rotations, and magnification. They map lines and circles to lines and circles, preserving angles and maintaining the same geometric structure.

Bi-Linear Transformations (Möbius Transformations): Möbius transformations are defined by functions of the form $f(z) = \frac{az+b}{cz+d}$, where a, b, c, and d are complex constants with $ad - bc \neq 0$. Möbius transformations map circles and lines to circles and lines, preserving angles. They are particularly useful in the study of complex analysis, geometry, and differential equations.

Inverse Transformations: These are the simplest conformal mappings, defined by functions of the form $f(z) = \frac{1}{z}$, where $z \neq 0$ isare complex constants. Inverse transformations maps lines and circles to circles and lines.

Exponential and Logarithmic Transformations: These transformations involve functions such as $f(z) = e^z$ and $f(z) = log(z)$, where e is Euler's number and $log(z)$ represents the complex logarithm. Exponential transformations map horizontal lines to spirals in the complex plane, while logarithmic transformations map regions onto strips.

Trigonometric Functions: Sine, cosine, and other trigonometric functions can be extended to the complex plane. Conformal mappings involving trigonometric functions often arise in the study of harmonic functions and potential theory.

Schwarz-Christoffel Transformations: These are specific conformal mappings that map the upper half-plane to polygonal regions. They are defined by elliptic integrals and are particularly useful in solving problems involving polygonal domains.

Theorem

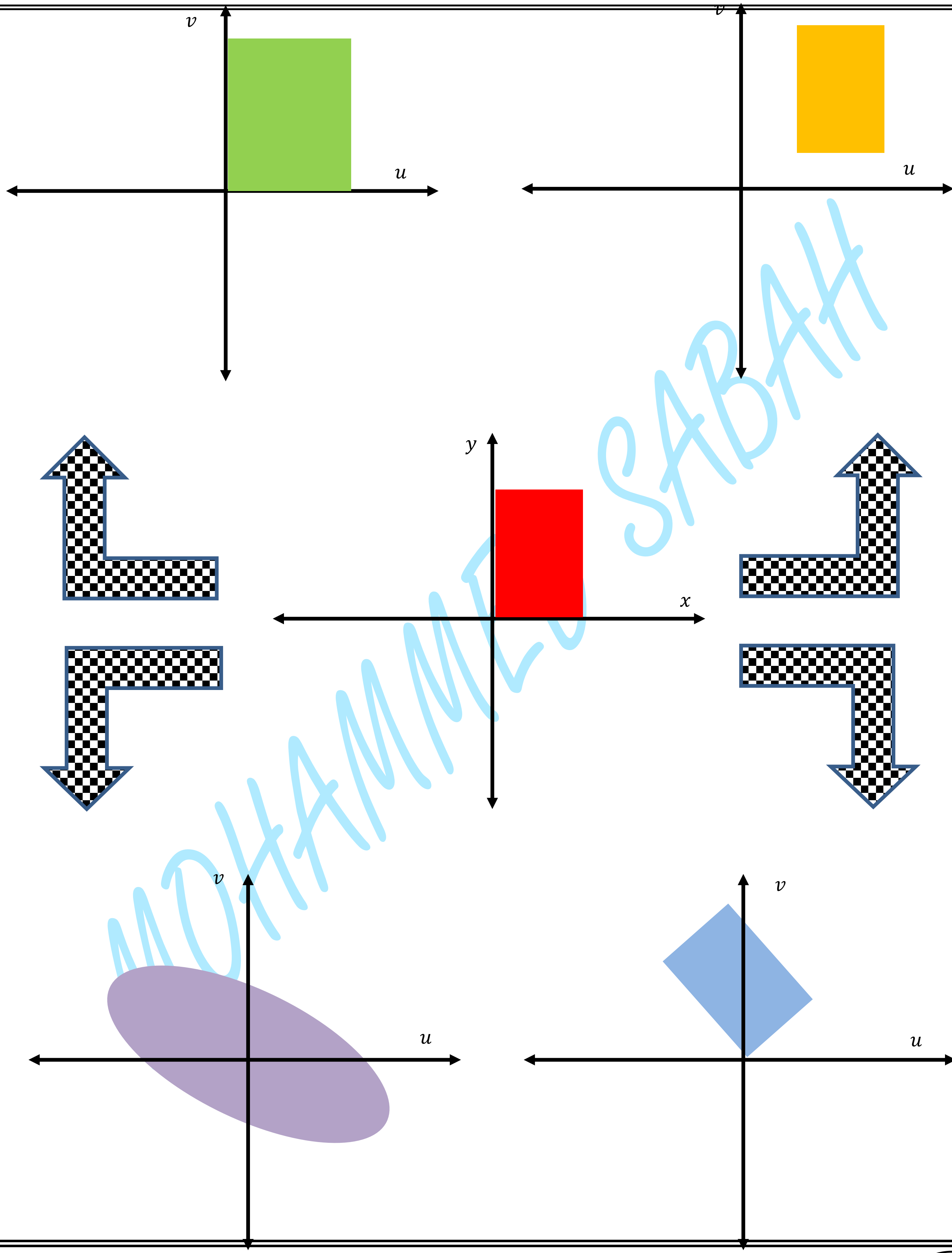
- if $f(z)$ is analytic then the mapping is conformal .
- if $f(z)$ is analytic function the $f(z)$ must be conformal at the points when $f'(z) = 0$
- if $f(z)$ is analytic function and $f'(z) \neq 0$ in region of a $z - plane$, then the mapping $w = f(z)$ is conformal at all points in a region.

Transformation

for every point (x, y) in z-plane the relation $w = f(z)$ difines a corresponding point (u, v) in the w-plane , we call this transformation or a mapping of z-plane into w-plane if a point z_0 maps into the point w_0 , such that w_0 is also known as the image of z_0 .

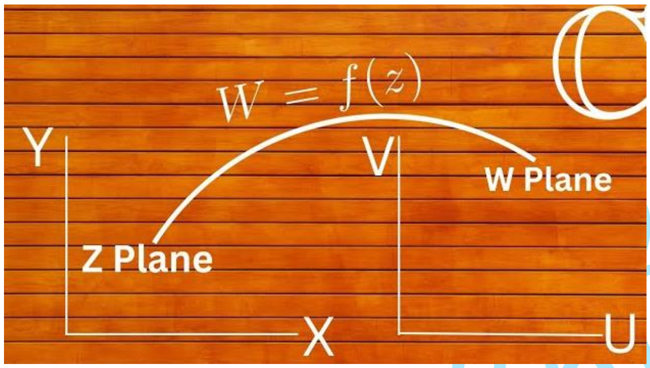
Some elementary transformation

1. Transition 2. Rotation 3. Magnification 4. Inversions


v
u
v
u
y
x
v
u
v
u

1. Transition

A complex transition is a map from z-plane (x, y) to w-plane (u, v) of the form $f: z \to z + \alpha$, $f(z) = z + \alpha$ where α is a fixed complex number ,which corresponding to transition (shifting) of point the complex number α



Example

What is the image of the rectangle region of the $z - plane$ Bounded by the lines $x = 0$, $y = 0$, $x = 1$, $y = 2$ under the Transformation $w = z + (2 - i)$ in the $w - plane$.

Solution

Let $w = u + iv$, $z = x + yi$

$w = z + 2 - i$

$u + iv = x + yi + 2 - i$

$u + iv = x + 2 + i(y - 1)$

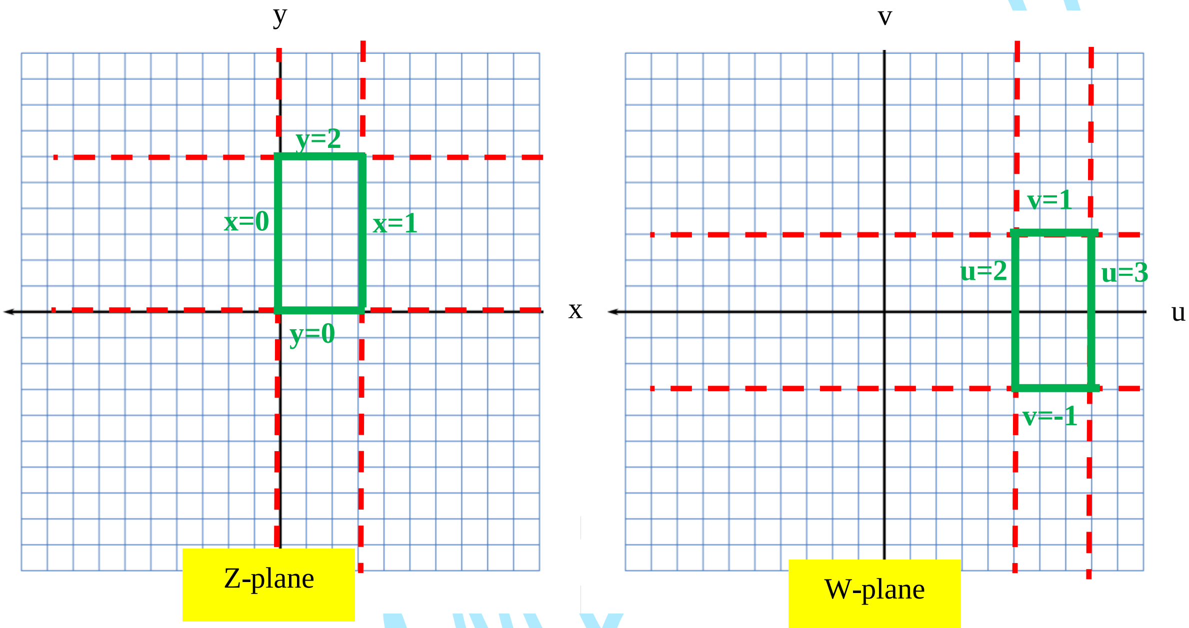
$u = x + 2$, $v = y - 1$

<u>z-plane</u>		<u>w-plane</u>		
$x = 0$	$\rightarrow$	$u = 0 + 2$	$\rightarrow$	$u = 2$

$y = 0 \quad \rightarrow \quad v = 0 - 1 \quad \rightarrow \quad v = -1$

$x = 1 \quad \rightarrow \quad u = 1 + 2 \quad \rightarrow \quad u = 3$

$y = 2 \quad \rightarrow \quad v = 2 - 1 \quad \rightarrow \quad v = 1$



Example

what the image of triangle region of the $z - plane$ bounded by the lines $x = 0$, $y = 0$, $x + y = 1$,under the transformation

$w = z + (1 + i)$ *in the* $w - plane$.

Solution

$w = z + (1 + i)$, $\quad w = u + vi$, $\quad z = x + yi$

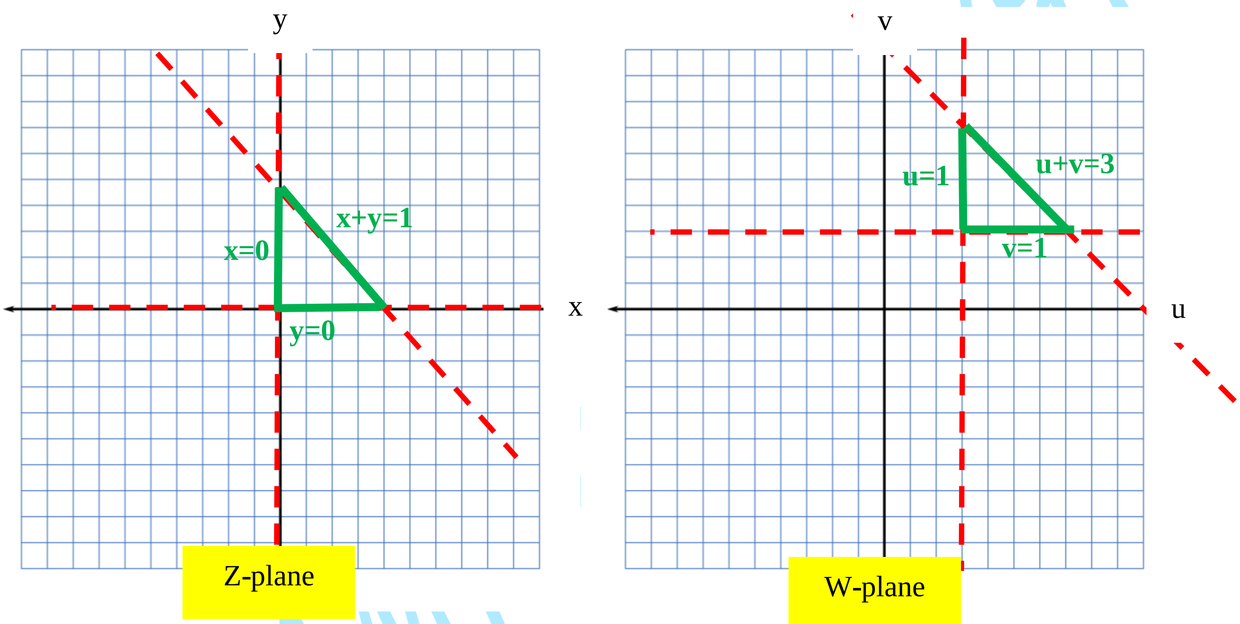
$u + iv = x + yi + 1 + i$

$$u + iv = (x+1) + (y+1)i \rightarrow \begin{cases} u = x+1 \\ v = y+1 \end{cases}$$

$$x = 0 \rightarrow u = 0+1 \rightarrow u = 1$$

$$y = 0 \rightarrow v = 0+1 \rightarrow v = 1$$

$$x + y = 1 \rightarrow u - 1 + v - 1 = 1 \rightarrow u + v = 3$$



2. Magnification

A complex magnification is a map from z-plane to w-plane of the from $z \rightarrow \alpha z$ where $\alpha \epsilon\ R^{++}$ which corresponds to magnification about the origin of points in the complex plane by the factor α , $if\ \alpha > 1$ the figure is largest but when $\alpha < 1$ the figure smallest

Example

What is the image of the rectangle region of the z − plane bounded by the lines $x = 0$, $y = 0$, $x = 1$, $y = 2$ under the transformation : $w = 2z$ in the w-plane .

Solution

$w = 2z\,, w = u + iv\,, z = x + yi\,, \alpha = 2 > 1$

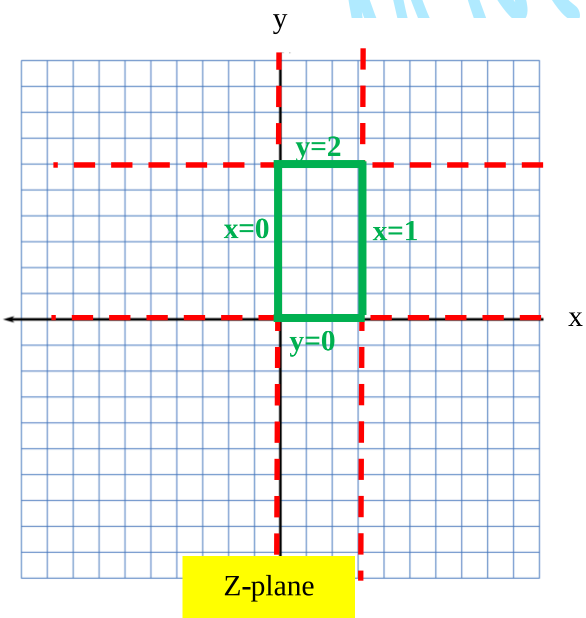
$u + vi = 2(x + yi)$

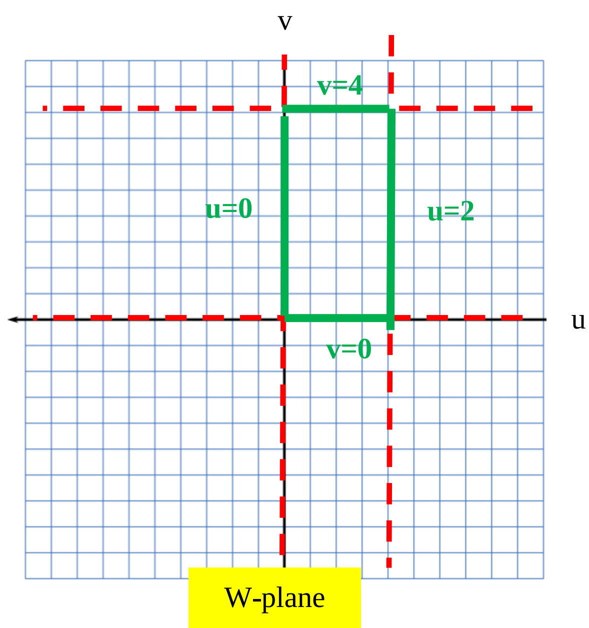
$u + vi = 2x + 2yi \;\rightarrow\; \begin{cases} u = 2x \\ v = 2y \end{cases}$

$x = 0 \;,\; y = 0 \;,\; x = 1 \;,\; y = 2$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$u = 0 \;,\; v = 0 \;,\; u = 2 \;,\; v = 4$





Example

what is the image of the triangle region in $z-plane$ bounded by the lines $x=0$, $y=0$, $x+y=2$ under the transformation $w=\frac{1}{2}z$

Solution

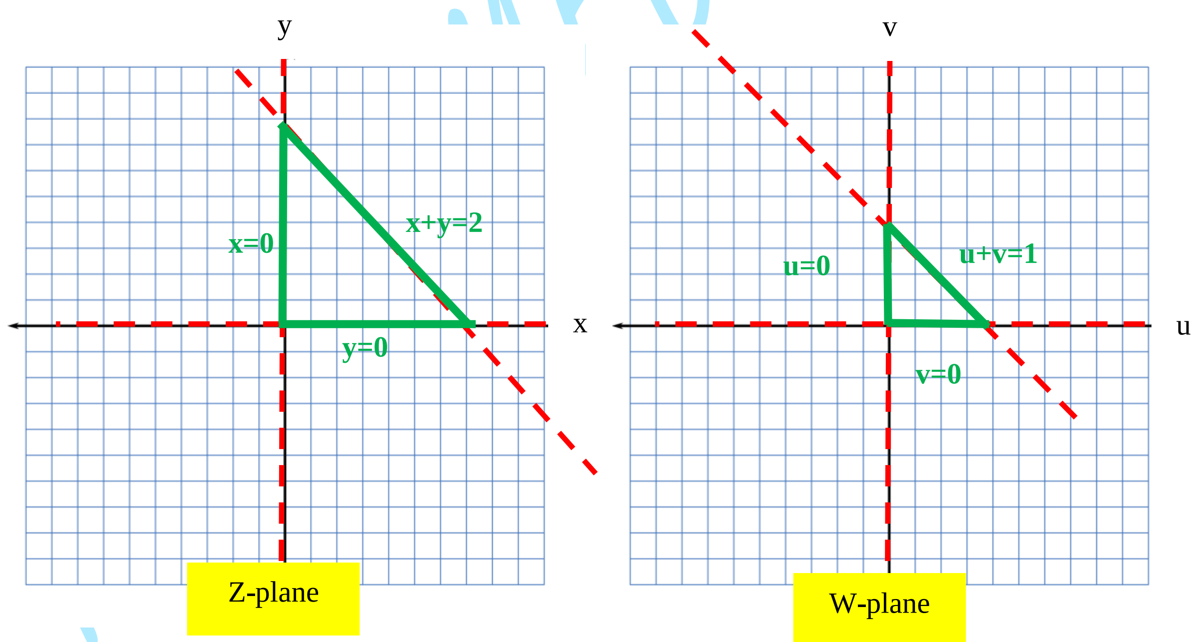
$w=\frac{1}{2}z$, $w=u+iv$, $z=x+yi$, $\alpha=\frac{1}{2}<1$

$u+vi=\frac{1}{2}(x+yi) \rightarrow u=\frac{1}{2}x$, $v=\frac{1}{2}y$

$x=0 \rightarrow u=0$

$y=0 \rightarrow v=0$

$x+y=2 \rightarrow 2u+2v=2 \rightarrow u+v=1$



3. Rotation

a complex rotation is a map from z-plane to ew-plane of the from $z \to e^{i\alpha} z$, $\alpha \in R$,or $f(z) = e^{i\alpha} z$ which corresponds to countor Clockwise rotation by θ radians about the origin of points in the complex plane

$$w = e^{i\alpha} z \quad \to w = e^{i\alpha} r e^{i\theta} = r e^{i(\theta+\alpha)}$$

note : when $\alpha > 0$ the rotation is countor-Clockwise , and when $\alpha < 0$ the rotation is clockwise .

Example

what is the image of the rectangular region of the $z - plane$ bounded By the lines $x = 0$, $y = 0$, $x = 1$, $y = 2$ under the trans formation $w = e^{i\frac{\pi}{4}} z$

Solution

$$w = e^{i\frac{\pi}{4}} z$$

$$u + vi = \left(cos\frac{\pi}{4} + isin\frac{\pi}{4}\right)(x + yi)$$

$$u + vi = \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)(x + yi)$$

$$\sqrt{2}\,u + i\sqrt{2}\,v = x + ix + iy - y$$

$$\sqrt{2}\,u = x - y \quad \dots(1)$$

$$\sqrt{2}\,v = x + y \quad \dots(2)$$

$$\sqrt{2}(u + v) = 2x \quad \to u + v = \sqrt{2}\,x$$

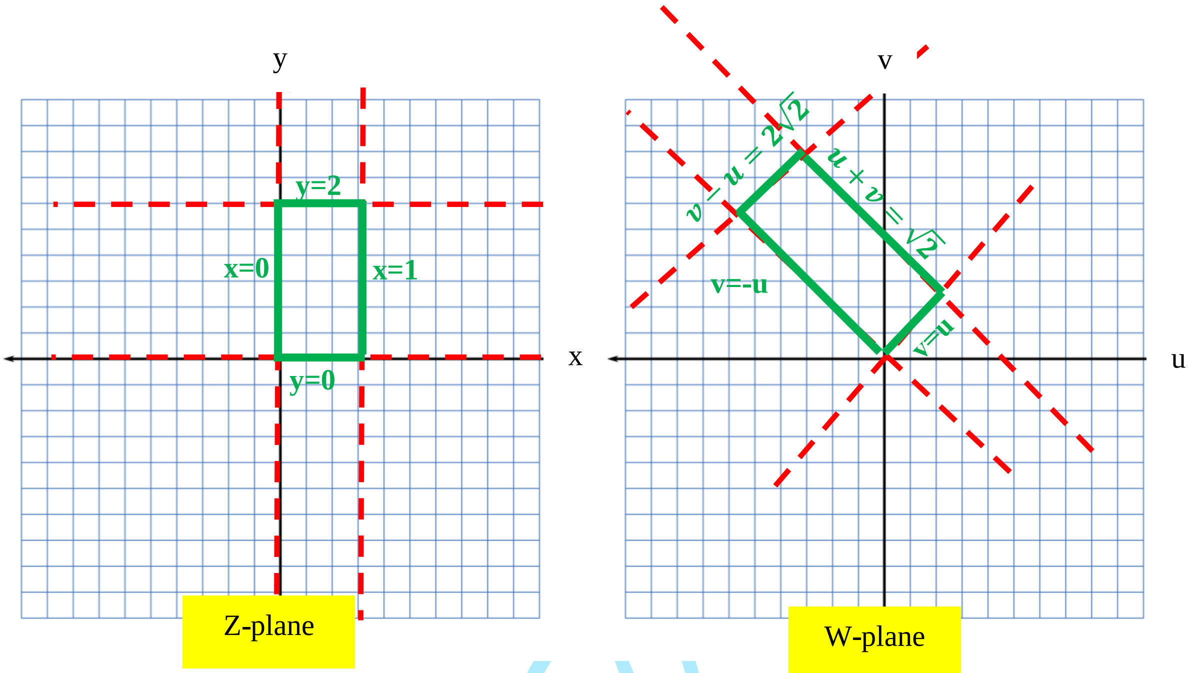
$$\sqrt{2}(v - u) = 2y \quad \to v - u = \sqrt{2}\,y$$

$$x = 0 \quad \to \quad u + v = 0 \quad \to \quad v = -u$$

$$y = 0 \quad \to \quad v - u = 0 \quad \to \quad v = u$$

$$x = 1 \quad \to \quad u + v = \sqrt{2}$$

$y = 2 \quad \to \ v - u = 2\sqrt{2}$



Example

what is the image of the triangular region of the $z - plane$ bounded By the lines $x = 0$, $y = 0$, $\sqrt{3}\,x + y = 1$ under the trans formation $w = e^{i\frac{\pi}{3}}\, z$

Solution

$let \quad w = u + iv \ , \quad z = x + yi \ , \qquad \alpha > 0$

$w = e^{i\frac{\pi}{3}}\, z \quad \to \ u + iv = \left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right)(x + yi)$

$u + vi = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)(x + yi)$

$2\,u + 2\,vi = x + \sqrt{3}xi + yi - \sqrt{3}y$

$2\,u = x - \sqrt{3}y \quad \dots(1)$

$2\,v = \sqrt{3}x + y \quad \dots(2) \qquad * \sqrt{3}$

$2u = x - \sqrt{3}y \quad \dots(1)$

$2\sqrt{3}v = 3x + \sqrt{3}y \quad \dots(2) \qquad sum$

$2\left(u + \sqrt{3}v\right) = 4x \;\rightarrow\; u + \sqrt{3}v = 2\,x \quad \dots(3)$

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multiply (1) by -3 and sum (1) with (2)

$-6u = -3x + 3\sqrt{3}y \quad \dots(1)$

$2\sqrt{3}v = 3x + \sqrt{3}y \quad \dots(2)$

$2\sqrt{3}\,v - 6u = 4\sqrt{3}y \quad \div 2\sqrt{3}$

$v - \sqrt{3}u = 2y \quad \dots(4)$

$u + \sqrt{3}v = 2\,x \quad \dots(3)$

<u>z-plane w-plane</u>

$x = 0 \rightarrow u + \sqrt{3}v = 0 \rightarrow v = \frac{-1}{\sqrt{3}}u$

$y = 0 \rightarrow v - \sqrt{3}u = 0 \rightarrow v = \sqrt{3}u$

$\sqrt{3}x + y = 1 \rightarrow 2v = 1 \rightarrow v = \frac{1}{2}$