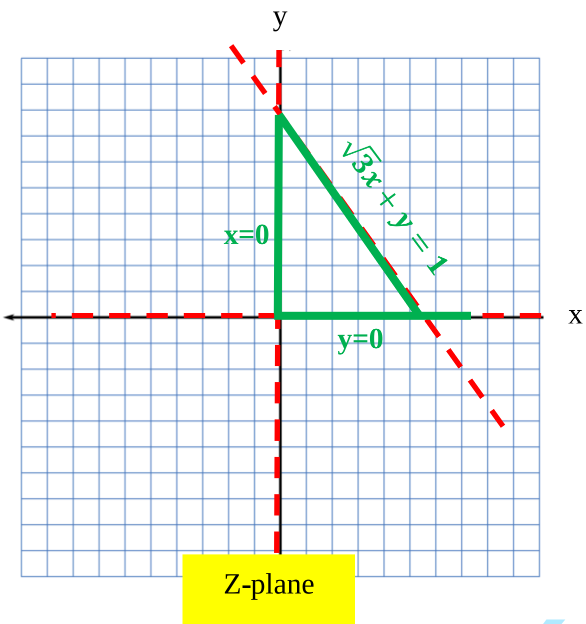
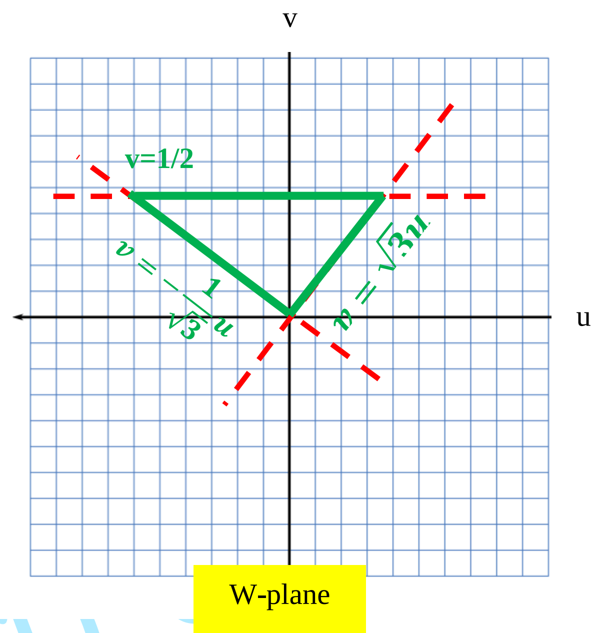




4. Inversion

a complex transformation inversion is a map from z-plane to w-plane of the from $z \to \frac{1}{z}$ or $f(z) = \frac{1}{z}$, where z is a complex variable , when the number z having modulus is r and argument is θ ,the inverse $\frac{1}{z}$ having modulus $\frac{1}{r}$ and argument $(-\theta)$

$$z = x + yi \quad , z = re^{i\theta} \quad \begin{cases} mod = r \\ arg = \theta \end{cases}$$

$$w = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} \quad \begin{cases} mod = \frac{1}{r} \\ arg = -\theta \end{cases}$$

Example

find the image of the circle $x^2 + y^2 = 4y$ under the trans formation $w = \frac{1}{z}$

Solution

$$let\ w = \frac{1}{z} \quad , z = x + yi \quad , w = u + iv$$

$$u + vi = \frac{1}{x + yi} \quad \rightarrow \quad x + yi = \frac{1}{u + iv} \cdot \frac{u - iv}{u - iv}$$

$$x + yi = \frac{u - iv}{u^2 + v^2} \quad , \quad x + yi = \frac{u}{u^2 + v^2} - i\frac{v}{u^2 + v^2}$$

$$x = \frac{u}{u^2 + v^2} \qquad y = \frac{-v}{u^2 + v^2}$$

$$now: \ x^2 + y^2 = 4y$$

$$\frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} = \frac{-4v}{u^2 + v^2}$$

$$\frac{u^2+v^2}{(u^2+v^2)^2} = \frac{-4v}{u^2+v^2} \quad \rightarrow -4v = 1 \quad \rightarrow v = \frac{-1}{4} \quad \text{[in w-plane]}$$

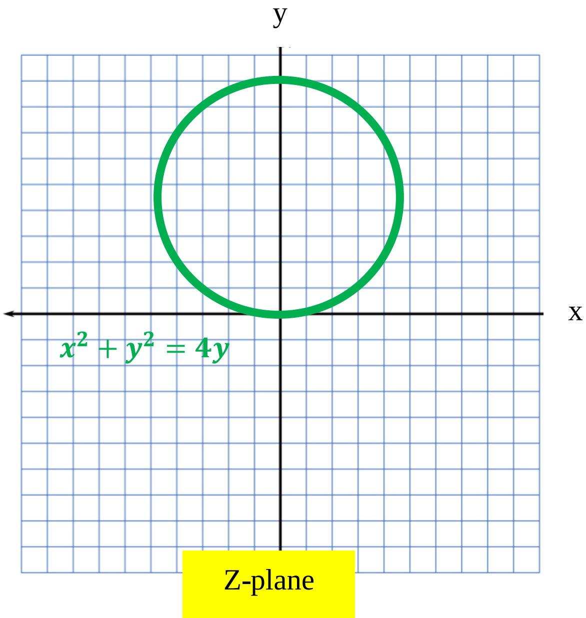
to graph the function in z-plane

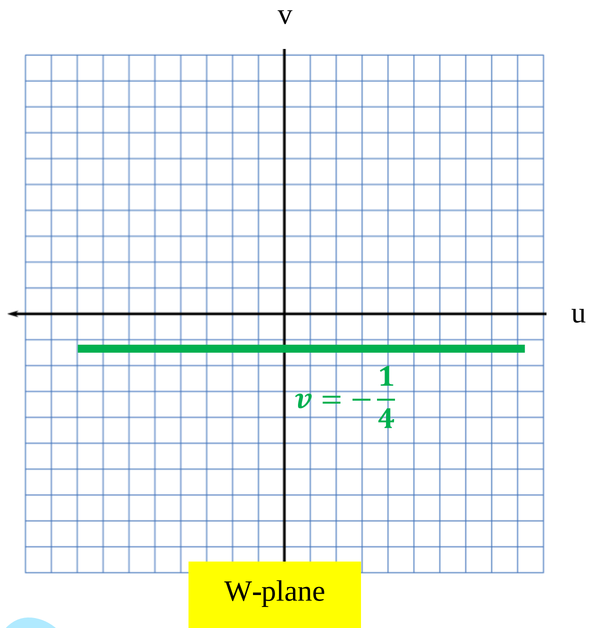
$$x^2 + y^2 = 4y$$

$$x^2 + y^2 - 4y = 0$$

$$x^2 + y^2 - 4y + 4 = 0 + 4$$

$x^2 + (y - 2)^2 = 4$, is circle with center is (0,2) and radius is 2





Example

Find the image of the line $y - x + 1 = 0$ under the inverse trans formation.

Solution

$$let\ w = \frac{1}{z} \quad ,z = x + yi \quad ,w = u + iv$$

$$u + vi = \frac{1}{x + yi} \rightarrow \quad x + yi = \frac{1}{u + iv} \cdot \frac{u - iv}{u - iv}$$

$$x + yi = \frac{u - iv}{u^2 + v^2} \quad ,x + yi = \frac{u}{u^2 + v^2} - i\frac{v}{u^2 + v^2}$$

$$x = \frac{u}{u^2 + v^2} \qquad\qquad y = \frac{-v}{u^2 + v^2}$$

$$now: \quad y - x + 1 = 0$$

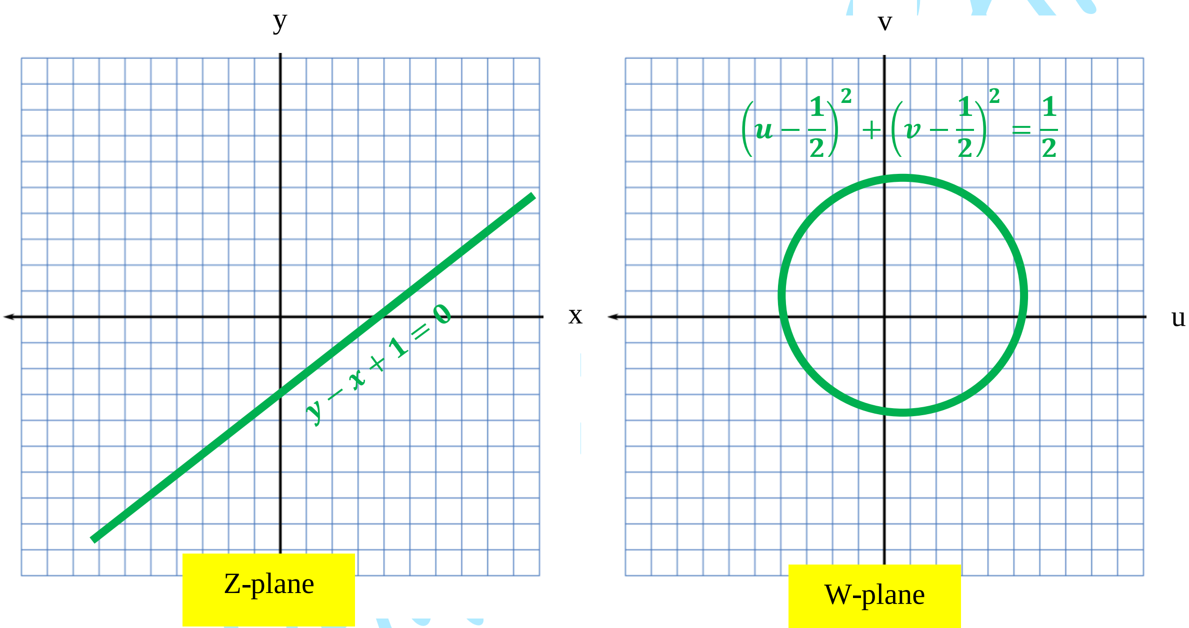
$$\frac{-v}{u^2 + v^2} - \frac{u}{u^2 + v^2} + 1 = 0$$

$-v-u+u^2+v^2=0$

$u^2-u+\frac{1}{4}+v^2-v+\frac{1}{4}=\frac{1}{4}+\frac{1}{4}$

$\left(u-\frac{1}{2}\right)^2+\left(v-\frac{1}{2}\right)^2=\frac{1}{2}$

the graph in w-plane represent a circle with center is $(\frac{1}{2},\frac{1}{2})$ and radius is $\frac{1}{\sqrt{2}}$



Example

Find the image of the $\infty-strip$ $\frac{1}{4}\leq y\leq\frac{1}{2}$ under the transformation $w=\frac{1}{z}$

Solution

$w=\frac{1}{z}$, $w=u+iv$, $z=x+yi$

$u+iv=\frac{1}{x+yi}\longrightarrow x+yi=\frac{1}{u+iv}\cdot\frac{u-iv}{u-iv}$

$x + yi = \frac{u - iv}{u^2+v^2} \rightarrow \begin{cases} x = \frac{u}{u^2+v^2} \\ y = \frac{-v}{u^2+v^2} \end{cases}$

$y = \frac{1}{4} \longrightarrow \frac{-v}{u^2+v^2} = \frac{1}{4} \longrightarrow u^2 + v^2 = -4v$

$u^2 + v^2 + 4v + 4 = 0 + 4$

$(u)^2 + (v+2)^2 = 4$

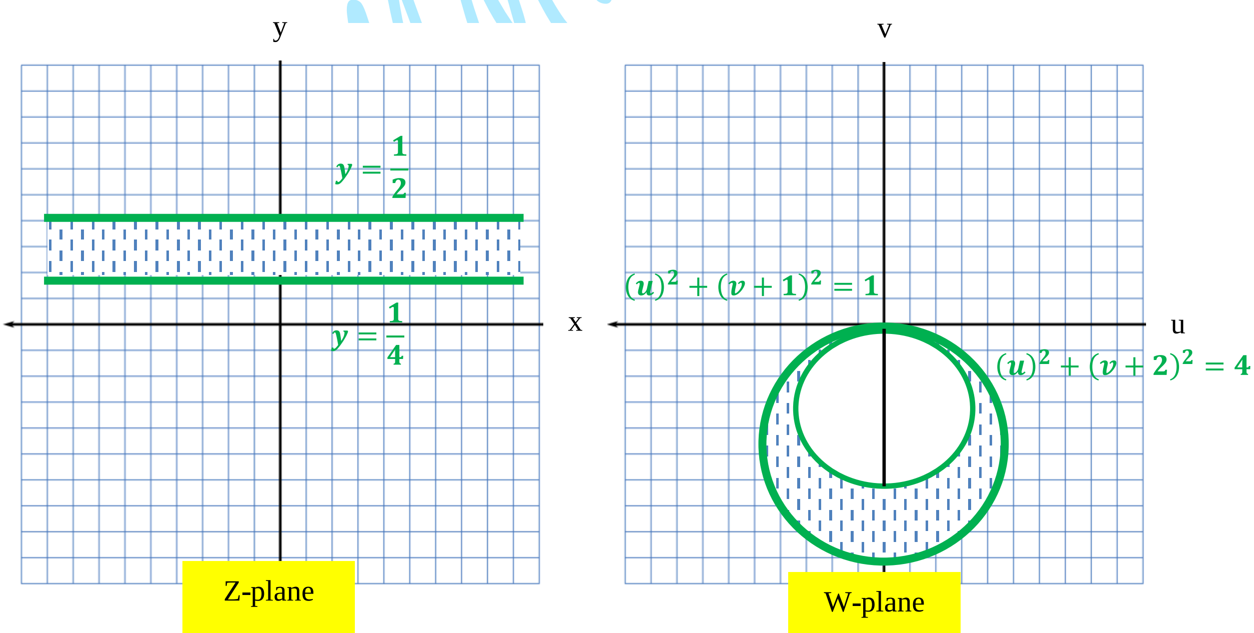
the eq. $y = \frac{1}{4}$ in z-plane represent a circle with center (0,-2) and radius is 2 in w-plane

$y = \frac{1}{2} \longrightarrow \frac{-v}{u^2+v^2} = \frac{1}{2} \longrightarrow u^2 + v^2 = -2v$

$u^2 + v^2 + 2v + 1 = 0 + 1$

$(u)^2 + (v+1)^2 = 1$

the eq. $y = \frac{1}{2}$ in z-plane represent a circle with center (0,-1) and radius is 1 in w-plane

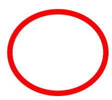
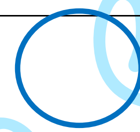
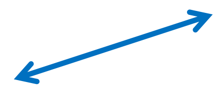
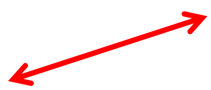
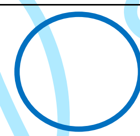


Theorem 1

the transformation $w = \frac{1}{z}$ maps the circle in z − plane to a circle in w − plane or a straight line if the circle passes at origin .

Theorem 2

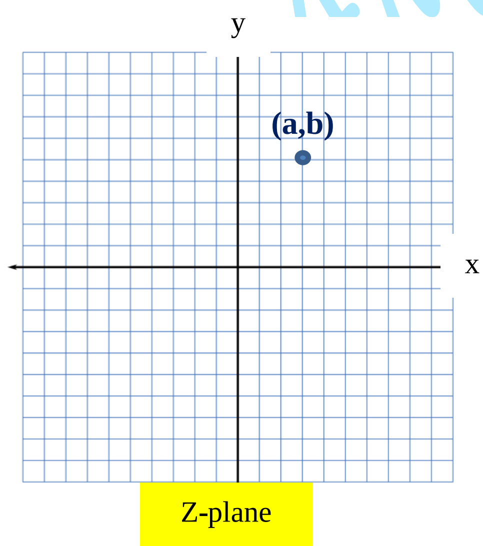
the transformation $w = \frac{1}{z}$ maps a line in z − plane to a circle or straight line in w − plane

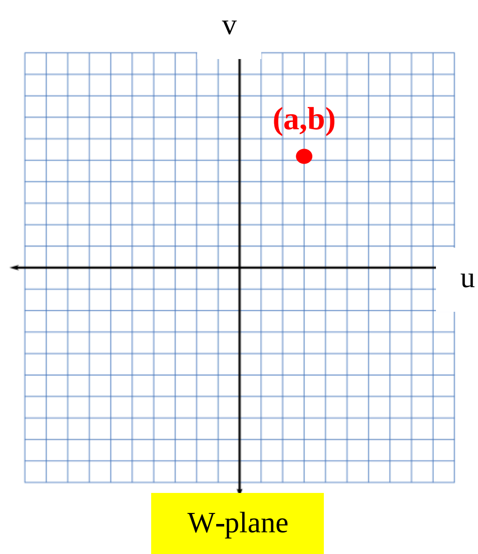
z-plane	w-plane
	 or 
	 or 

Invariant point (fixed)

its point which coincide with their transfc y tion are called fixed or invariant points

$w = f(z) = z$





Example

find the invariant point to

Solution

1. $w = \frac{z-1}{z+1}$

$\because w = f(z) = z$

$z = \frac{z-1}{z+1} \rightarrow z^2 + z = z - 1$

$z^2 = -1 \rightarrow z = \mp i$

To check sub. $(i, -i)$ in $w = \frac{z-1}{z+1}$

2. $w = \frac{3z-4}{z-1}$

$\because w = f(z) = z$

$z = \frac{3z-4}{z-1} \rightarrow z^2 + z = 3z - 4$

$z^2 - 4z + 4 = 0 \rightarrow (z-2)^2 = 0 \rightarrow z = 2$

Linear Transformation

a combination of transformations! Transition , Rotation and magnification is called Linear transformation , and the form of it :

$w = \alpha z + \beta$ where $\alpha, \beta \in \mathbb{C}$

Example

consider the linear transformation $w = (1+i)z + (2-i)$, determine and sketch the region in $w - plane$ into which the rectangular region bounded by the lines $x = 0$, $y = 0$, $x = 1$, $y = 2$, in the $z - plane$ is mapped

Solution

let $w = u + iv \quad z = x + yi$

$w = (1+i)z + (2-1)$

$u + iv = (1+i)(x+yi) + (2-1)$

$= x + yi + xi - y + 2 - i$

$u + iv = x - y + 2 + i(x + y - 1)$

$\boxed{u = x - y + 2} \qquad \boxed{v = x + y - 1}$

$u = x - y + 2 \quad \dots\dots\dots(1)$

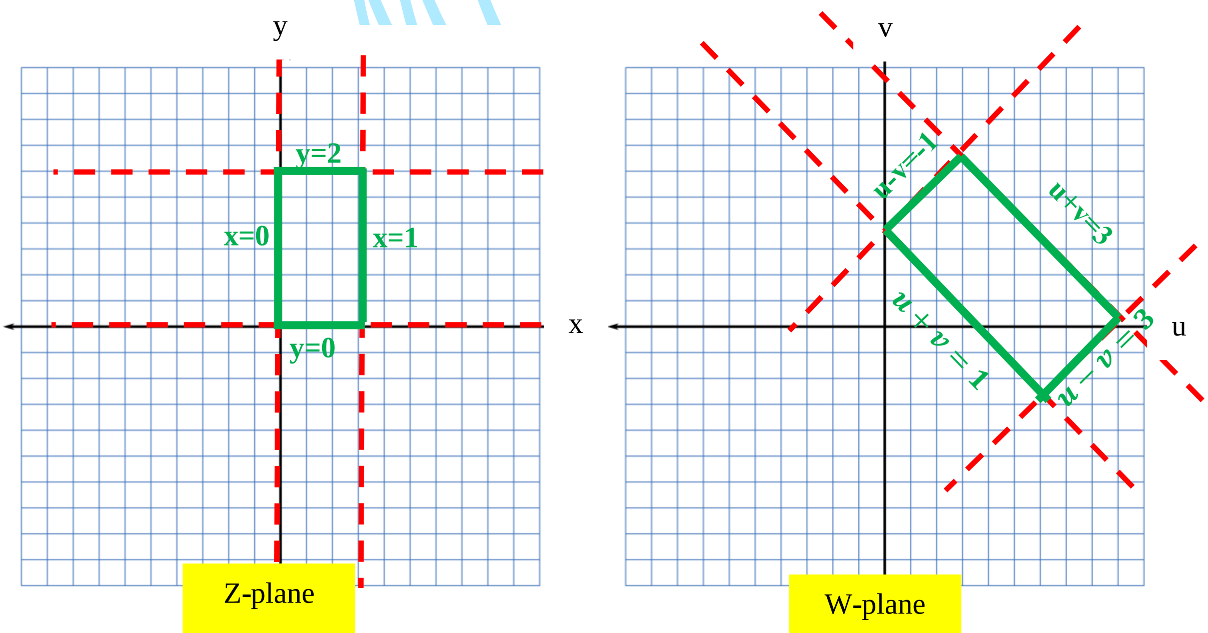
$v = x + y - 1 \quad \dots\dots\dots(2)$

$u + v = 2x + 1 \quad \dots\dots.(3)$ [by add 1 to 2]

$u - v = 3 - 2y \quad \dots\dots.(4)$ [by difference 2 from 1]

$x - plane: \quad x = 0 \quad , \quad y = 0 \quad , \quad x = 1 \quad , \quad y = 2$

$z - plane: \quad u + v = 1 \quad , \quad u - v = 3 \quad , \quad u + v = 3 \quad , \quad u - v = -1$



Example

Determine the linear transformation that map the rectangle

$z_1 = 0$, $z_2 = \frac{1}{2}$, $z_3 = i$, $z_4 = \frac{1}{2} + i$ **Onto the rectangle**

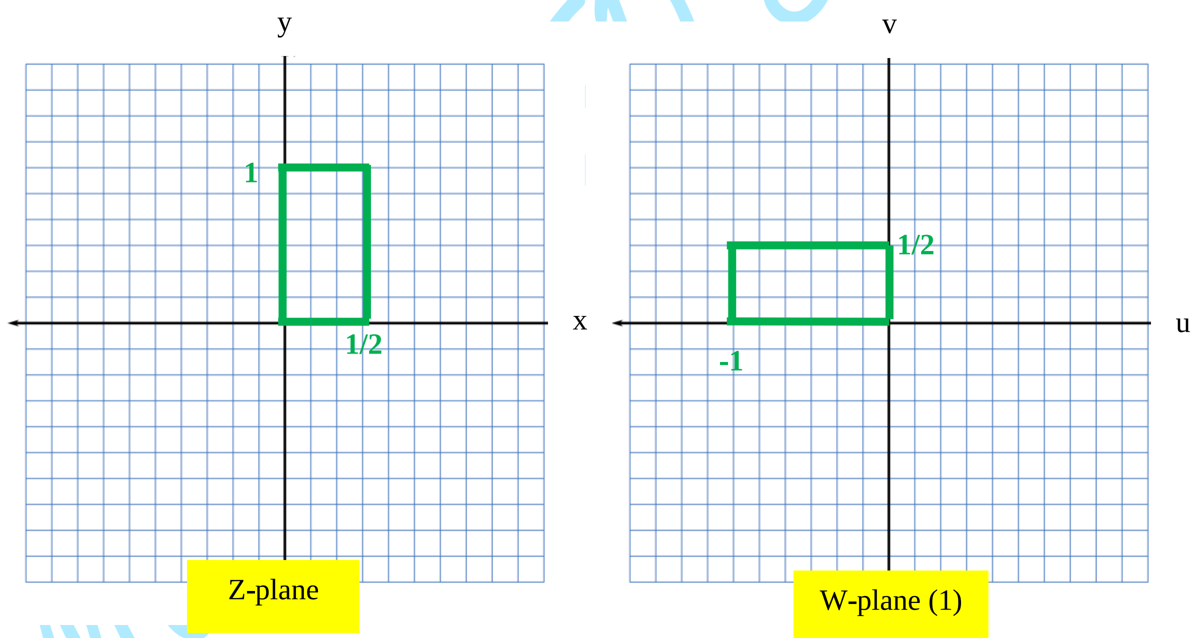
$w_1 = i$, $w_2 = 2i$, $w_3 = -2 + i$, $w_4 = -2 + 2i$

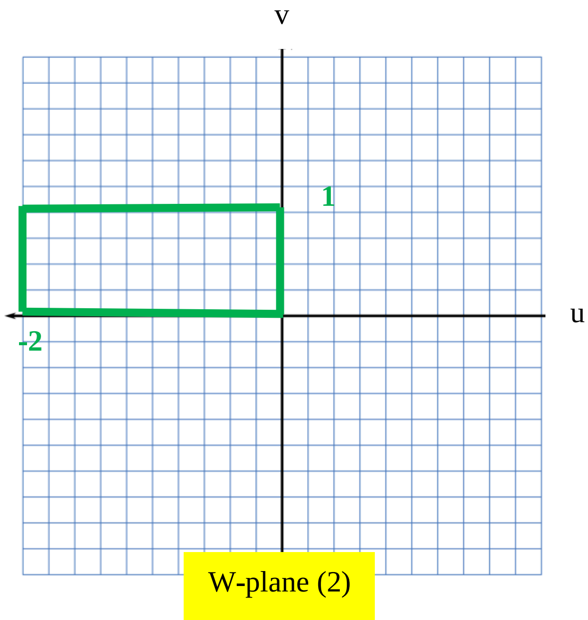
Solution

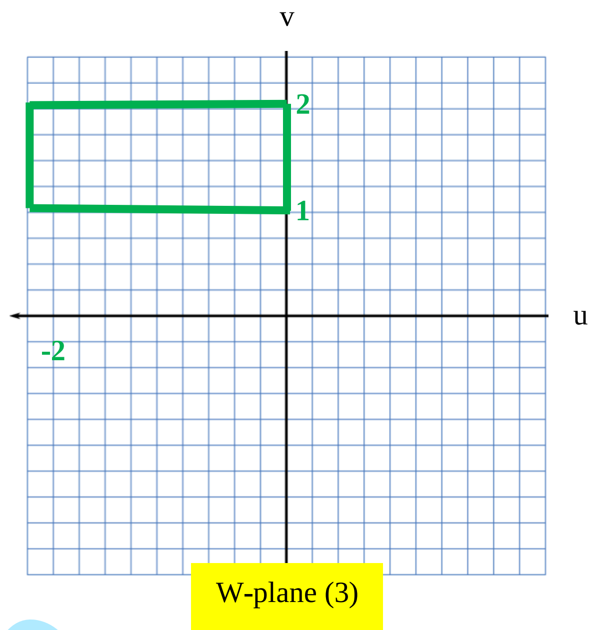
$z - plane$: $z_1(0,0)$, $z_2\left(\frac{1}{2},0\right)$, $z_3(0,1)$, $z_4\left(\frac{1}{2},1\right)$

$w - plane$: $w_1(0,1)$, $w_2(0,2)$, $w_3(-2,1)$, $w_4(-2,2)$

$z - plane$: $u + v = 1$, $u - v = 3$, $u + v = 3$, $u - v = -1$







$f_1 : z - plane \ \rightarrow w - plane(1)$, such that $f_1(z) = e^{i\frac{\pi}{2}} z = iz$

$f_2 : w - plane(1) \ \rightarrow w - plane(2)$, such that $f_2(z) = 2(iz) = 2iz$

$f_3 : w - plane(2) \ \rightarrow w - plane(3)$, such that $f_3(z) = (2iz) + i$

Example

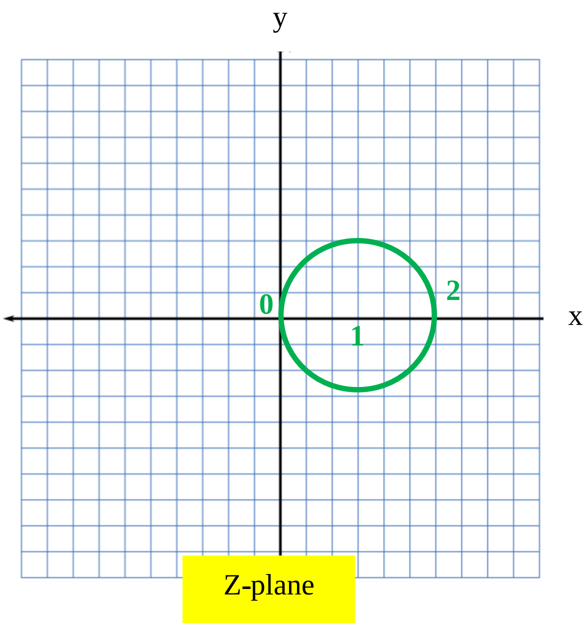
Determine the linear transformation that map the circle $C_1 : |z - 1| = 1$ in z-plane onto the circle $C_2 : \left|z - \frac{3}{2} i\right| = 2$ on w-plane

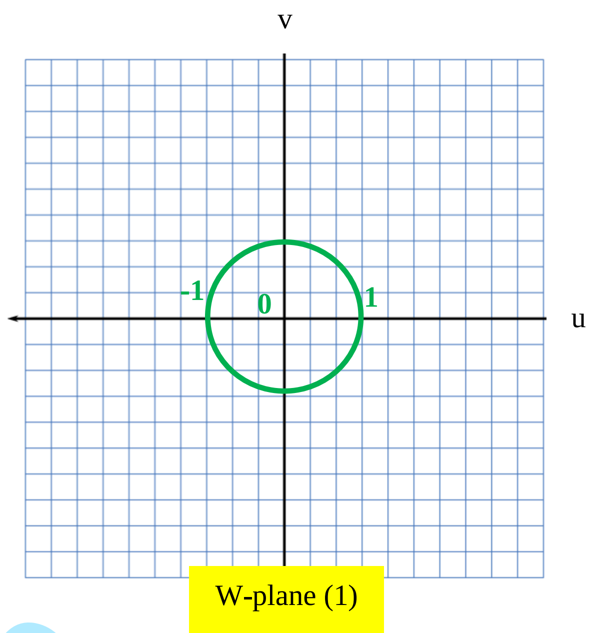
Solution

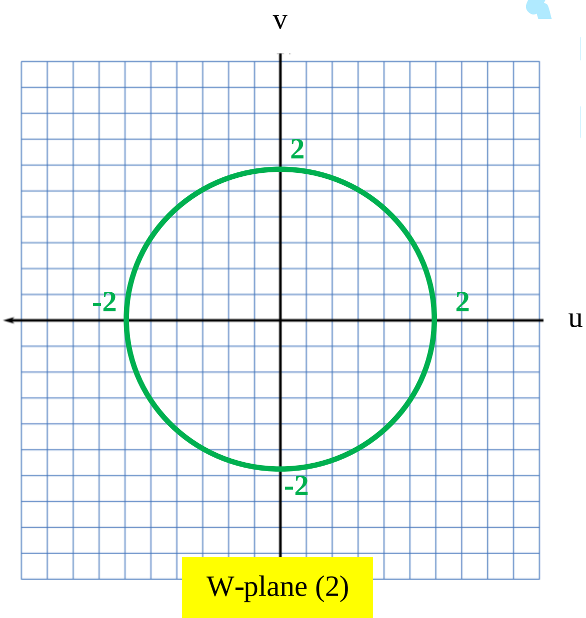
C_1 is represent a circle with center is $(1,0)$ and radius is 1

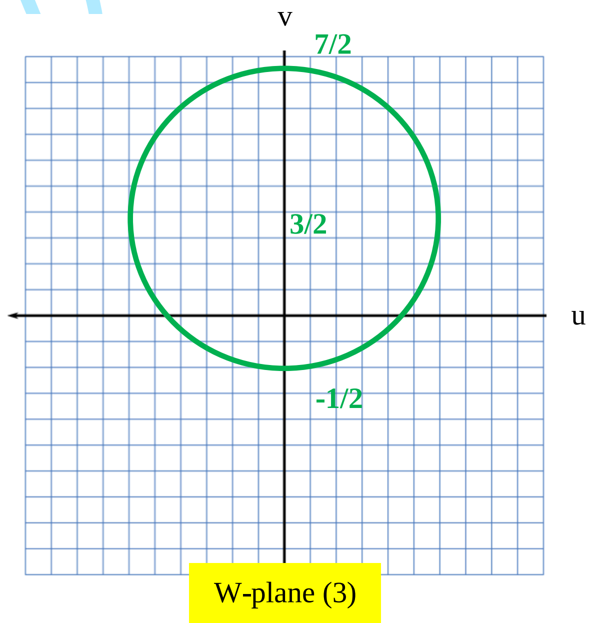
while

C_2 is represent a circle with center is $(0 , \frac{3}{2})$ and radius is 2









$f_1 : z - plane \ \rightarrow w - plane(1)$, such that $f_1(z) = z - 1$

$f_2 : w - plane(1) \ \rightarrow w - plane(2)$, such that $f_2(z) = 2(z - 1) = 2z - 2$

$f_3 : w - plane(2) \ \rightarrow w - plane(3)$, such that $f_3(z) = (2z - 2) + \frac{3}{2}i$

Bi-linear transformation

a combination of a transformations : transation, rotation, magnification and inversion are called bilinear transformation.

Theorem

the transformation of the form $w = \frac{az+b}{cz+d}$ is called a bilinear or fractional or mobious transformation when $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$

Theorem

the bilinear transformation $w = \frac{az+b}{cz+d}$ is a combination of elementary transformation.

Theorem

if bilinear transformation transform the point z_1, z_2, z_3 in $z - plane$ to the point w_1, w_2, w_3 in the $w - plane$ then :

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Example

Show that the transformation $w = \frac{2z+3}{z-4}$ maps the circle $x^2 + y^2 - 4x = 0$ in the z-plane into the straight line $4u + 3 = 0$ in the w-plane .

Solution

$$w = \frac{2z+3}{z-4} \rightarrow w(z-4) = 2z+3$$

$$wz - 4w = 2z + 3$$

$$z(w-2) = 4w + 3 \rightarrow \boxed{z = \frac{4w+3}{w-2}} \qquad \ldots\ldots\ldots ❶$$

$$x^2 + y^2 - 4x = 0$$

$x^2 - 4x + 4 + y^2 = 0 + 4$

$(x-2)^2 + y^2 = 4$ this represent

$|z-2| = 2$ ……… ❷ sub. ❶ in ❷

$\left|\frac{4w+3}{w-2} - 2\right| = 2$

$\left|\frac{4w+3-2w+4}{w-2}\right| = 2 \rightarrow \left|\frac{2w+7}{w-2}\right| = 2$

$|2w+7| = 2|w-2|$, $w = u + iv$

$|2(u+iv)+7| = 2|u+iv-2|$

$|2u+7+2iv| = 2|u-2+iv|$

$\sqrt{(2u+7)^2 + (2v)^2} = 2\sqrt{(u-2)^2 + v^2}$ Square the two sides

$(2u+7)^2 + 4v^2 = 4((u-2)^2 + v^2)$

$4u^2 + 28u + 49 + 4v^2 = 4(u^2 - 4u + 4 + v^2)$

$4u^2 + 28u + 49 + 4v^2 = 4u^2 - 16u + 16 + 4v^2$

$44u + 33 = 0$ $\div 11$

$4u + 3 = 0$