Theory of Ordinary Differential Equations

Definition (1): A Differential Equation (D.E.) is an
تربط
equation connecting an independent variable x, dependent variable y, and it's derivatives, y', y'', ... etc i.e an equation of the form

$$f(x, y, y', y'', \dots, y^{(n)}) = 0$$

Definition (2): The Solution of D.E. is any relation
بين
among variables of D.E. and it has the following:

- No derivative
- Bounded in the interval
- Satisfy the D.E.

مسألة كوشي — مسألة القيمة الابتدائية

Definition (3): The Cauchy problem or initial value problem (I.V.P.) is a problem of finding the solution $y = y(x)$ of the D.E. $y' = f(x,y)$ satisfying the initial condition $y(x_0) = y_0$ i.e.

$$y' = f(x,y) \quad , \quad y(x_0) = y_0$$

نظريات الوجود والوحدانية للحل

The Existence and Uniqueness Theorems of Solution :

Existence كوشي

Theorem (1) [Cauchy - Peano theorem] نظرية الوجود لبيانو

Let a D.E. $y' = f(x,y)$. be given where the function

محددة

$f(x,y)$ is determinde in Some domain D in the xy-plane contaning a point (x_0, y_0) as following:

$$D = \{ (x,y) : |x - x_0| \leq a \ , \ |y - y_0| \leq b \}$$

then if the function $f(x,y)$ satisfies the following conditions :

a- $f(x,y)$ is continuous function of the variable x and y in D .

b- $f(x,y)$ is bounded by M i.e

$$|f(x,y)| \leq M$$

اصغر

c- h is smallest numbers $a, \frac{b}{M}$

i.e $\quad h = \min\left(a, \frac{b}{M}\right)$

Then an interval $I_h : |x - x_0| \leq h$ will be found Solution $y = \phi(x)$ of the given D.E. satisfying condition $y(x_0) = y_0$.

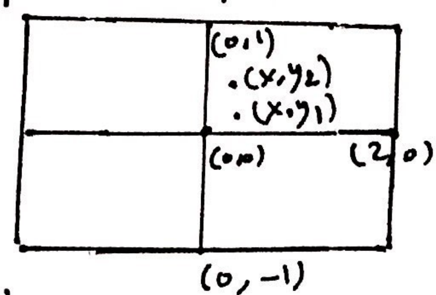
Definition (3): The function $f(x,y)$ defined on some domain D is said to satisfying in D, Lipschitz (ليبشتز) Condition (شرط) if there is constant L (Lipschitz Constant) (ثابت ليبشتز) such that for any y_1, y_2 in D and any x in D the inequality

$$|f(x,y_2)-f(x,y_1)| \leq L\,|y_2-y_1|$$

is satisfying.

Example (1): Given $S = \{(x,y) : |x| \leq 2 \,,\, |y| \leq 1\}$ and f defined in S as following $\forall (x,y)$ in S then $f(x,y) = x^2 y^2$, Explain wheather this function is satisfying Lipschitz condition in S and find Lipshitz constant.

Solution: Let $(x,y_1), (x,y_2)$ any two points is S, then



$$\begin{aligned}
|f(x,y_2)-f(x,y_1)| &= |x^2y_2^2 - x^2y_1^2| \\
&= x^2|y_2^2 - y_1^2| \\
&\leq x^2(|y_2|+|y_1|)\,|y_2-y_1| \\
&\leq 2^2(1+1)\,|y_2-y_1| \\
&\leq 8\,|y_2-y_1|
\end{aligned}$$

the fun. f is satisfying Lipschitz Condition with $L = 8$

Note: Lipschitz condition may be replaced by the bounded of partial derivative $\frac{\partial f}{\partial y}$.
يستبدل

Example (2): Given $f(x,y) = y^{2/3}$ $\forall\, x,y$ in R^2. Prove the function f no-satisfy Lipschitz condition in any region in R^2 contain the points of x-axis and prove $y=0$, $y=\frac{x^3}{27}$ are solutions of the problem.
لا تحقق

Solution: Consider S any region in R^2 contain the points of x-axis, since $\frac{\partial f}{\partial y} = \frac{2}{3} y^{-1/3} = \frac{2}{3}\frac{1}{\sqrt[3]{y}}$ and since S contain the points of x-axis only $(y=0)$ then $\frac{\partial f}{\partial y}$ discontinuous in these points and this means it is no-satisfies the conditions of above note
غير مستمرة
لا تحقق

i.e. f is not satisfy Lipschitz conditions in S.

it is clear $y=0$ is a solution of this problem (trivial solution).
حل تافه

to explain $y=\frac{x^3}{27}$ another solution of this problem we note:

$$y' = y^{2/3} = \left(\frac{x^3}{27}\right)^{2/3} = \frac{x^2}{9} \text{ and for } y=\frac{x^3}{27}$$

$$y' = 3\frac{x^2}{27} = \frac{x^2}{9} \Rightarrow \therefore L.H.S = R.H.S \quad \therefore y=\frac{x^3}{27} \text{ is}$$

another solution.

Theorem (2): (Cauchy - Peano theorem) مبرهنة الوجود والوحدانية

If the function $f(x,y)$ satisfies all conditions of Cauchy-Peano theorem (theorem 1) as well (بالإضافة) as lipchitz condition in y and in the domain D. then I.V.P. has unique (وحيد) solution (حل)

The method of sussessive approximations (Picard method)

طريقة التقريبات المتعاقبة (المتتالية)

suppose it is required (تتطلب) to find the solution $y = y(x)$ of D.E. $y' = f(x,y)$ satisfying the initial condition $y(x_0) = y_0$. the solution of this problem can be found by the method of successive approximation which is as following we construct (نركب) a sequence $\{g_n(x)\}$ of functions defined by the relations:

$$g_n(x) = y_0 + \int_{x_0}^{x} f(t, g_{n-1}(t))\, dt \quad , n > 1, 2, 3, \ldots$$

As zero-order approximation $g_0(x)$, we take function continuous in the neighbourhood (جوار) of the point (x_0, y_0) i.e.

$g_0 = y_0$

Example (3): Given $S = \{(x,y) : |x| \leq 3,\ |y-1| \leq 1$

and $y' = x^2 y^2,\ y(0) = 1 \quad \ldots (1)$

1) Is the I.V.P. (1) satisfy the conditions of Cauchy-Peano theorem.

2) If the answer "yes", find the interval I_h

3) find Picard approximations g_1, g_2, g_3

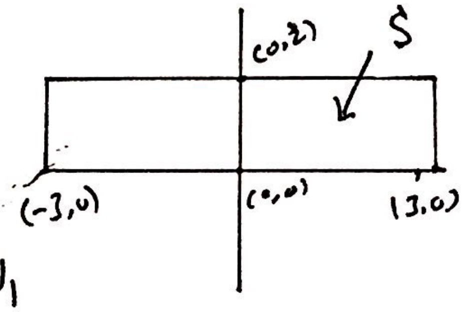
4) sketch the مخططات digrams for g_0, g_1, g_2, g_3

Solution:

(1)

1 - Since $f(x,y)$ is polynomail, then it is Continuous in $\mathbb{R}^2$

i.e. it is continuous in S



2 - from (1) we find

$$|f(x,y)| = x^2 y^2 \leq 3^2 \cdot 2^2 = 36 = M \text{ مثلا}$$

$\therefore$ the function is boundened is S

3 - If we take any two points $(x,y_1), (x,y_2)$ in S

then
$$|f(x,y_2) - f(x,y_1)| = |x^2 y_2^2 - x^2 y_1^2|$$
$$= x^2 |y_2^2 - y_1^2|$$
$$\leq x^2 (|y_2| + |y_1|) \cdot |y_2 - y_1|$$
$$= 3^2 (4|y_2 - y_1|) = 36\,|y_2 - y_1|$$

$\therefore \left| f(x, y_2) - f(x, y_1) \right| \leq 36 \left| y_2 - y_1 \right|$

this prove the function $f(x,y)$ satisfy Lipschitz condition with $L = 36$

$\therefore$ I.V.P. (1) satisfy all conditions of Cauchy-peano th.

② $I_h : |x - x_0| \leq h$

$$h = \min\left(a, \frac{b}{M}\right) = \min\left(3, \frac{1}{36}\right) = \frac{1}{36}$$

$\therefore I_h : |x| \leq \frac{1}{36}$

③ $g_n(x) = y_0 + \int_{x_0}^{x} f[t, g_{n-1}(t)]\, dt \qquad n = 1, 2, 3, \ldots$

when $n = 1$, $g_0 = y_0$ خطأ

$$g_1(x) = y_0 + \int_0^x f[t, g_1(t)]\, dt$$

$$= 1 + \int_0^x f[t, g_0(t)]\, dt$$

$$= 1 + \int_0^x t^2 \cdot g_0^2(t)\, dt = 1 + \int_0^x t^2 \cdot 1^2\, dt$$

$$= 1 + \int_0^x t^2\, dt = 1 + \left[\frac{t^3}{3}\right]_0^x = 1 + \frac{x^3}{3}$$

when $n = 2$

$$g_2(x) = y_0 + \int_0^x f[t, g_1(t)]\, dt$$

$$= 1 + \int_0^x t^2 \cdot g_1^2(t)\,dt = 1 + \int_0^x t^2 \cdot \left(1 + \frac{t^3}{3}\right)^2 \cdot dt$$

$$= 1 + \int_0^x t^2 \cdot \left(1 + \frac{2}{3}t^3 + \frac{1}{9}t^6\right) dt$$

$$= 1 + \int_0^x \left(t^2 + \frac{2}{3}t^5 + \frac{1}{9}t^8\right) dt$$

$$= 1 + \left[\frac{t^3}{3} + \frac{2}{3}\cdot\frac{t^6}{6} + \frac{1}{9}\cdot\frac{t^9}{9}\right]_0^x$$

$$\therefore g_2(x) = 1 + \frac{x^3}{3} + \frac{x^6}{9} + \frac{x^9}{81}$$

when $n = 3$

$$g_3(x) = y_0 + \int_0^x f[t, g_2(t)]\,dt$$

$$= 1 + \int_0^x t^2 \cdot \left(1 + \frac{t^3}{3} + \frac{1}{9}t^6 + \frac{1}{81}t^9\right)^2 dt$$

$$\therefore g_3(x) = 1 + \int_0^x \left(t^2 + \frac{t^5}{3} + \frac{t^8}{9} + \frac{t^{11}}{81} + \frac{t^5}{3} + \frac{t^2}{9} + \frac{t^4}{27} + \frac{t^{14}}{243} + \frac{t^8}{9} + \frac{t^{11}}{27} + \frac{t^{14}}{81} + \frac{t^{17}}{729} + \frac{t^{11}}{81} + \frac{t^{14}}{243} + \frac{t^{17}}{729} + \frac{t^{20}}{6561}\right) dt \;\cdots$$

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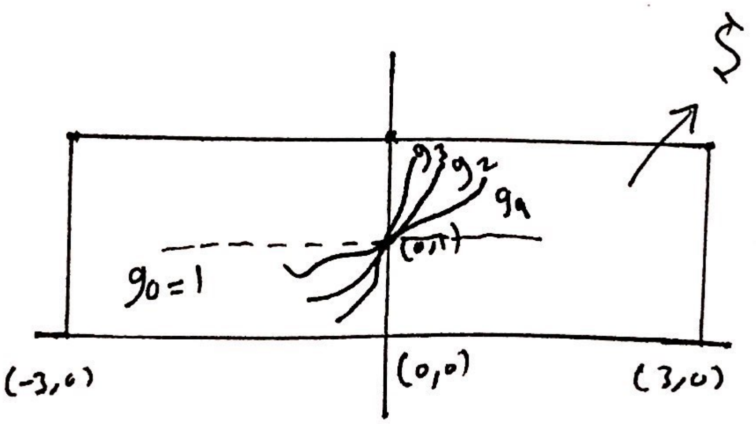

لايجاد العتيد M نلاحظات

$$M = \max |f(x,y)|$$

اي اعلى قيمة لـ f وعليه في حالتي الجمع والضرب نطبقه نفس الخطوات السابقة اي نأخذ اعلى قيمة لـ a, b مثلاً

اذا كانت

$$S = \{(x,y) : |x| \leq 2 \ , \ |y| \leq 3\}$$

$$f(x,y) = x^2 + y \Rightarrow M = 2^2 + 3 = 7$$

بينما في حالة الطرح نلاحظ ان كل دالة لها طريقة مختلفة لايجاد العتيد (اي اختيار a, b) والهدف منها جميعاً الحصول على اعلى قيمة للدالة $f(x,y)$ فمثلاً لتكن

$$S = \{(x,y) : |x| \leq 2 \ , \ |y| \leq 1\} \ , \ f(x,y) = x^2 - y^2 \ , \ y(0) = 0$$

في هذه الحالة نأخذ اعلى قيمة لـ x وهي 2 ونلاحظ ان قيم y هي $1, 0, -1$ وعليه لكي تكون الدالة اعلى ما يمكن نأخذ $y=0$ فتصبح $M = 2^2 - 0 = 4$ ولهذا $M = 4$.

<u>مثال اخر</u> ، لتكن

$$S = \{(x,y) : |x| \leq 2 \ , \ |y| \leq 1\}$$

$$f(x,y) = x^2 - xy$$

نلاحظ اعلى قيمة لـ x هي بينما قيم y هي $1, 0, -1$ وان قيمة y التي تجعل الدالة اعلى ما يمكن هي $y = -1$ ولهذا

$$M = 2^2 - 2(-1) = 4 + 2 = \underline{\underline{6}}$$

<u>مثال اخر</u> ،

$$S = \{(x,y) : |x| \leq 2 \ , \ |y| \leq 1\}$$

$$f(x,y) = x^2 - y + 3$$

نلاحظ عندما $y = -1$ نحصل على اعلى قيمة للدالة

$$M = 2^2 - (-1) + 3 = 8$$