Methods for Solving Ordinary Differential Equations

Fundamental Concepts مفاهيم اساسية

Definition (1): A Differential Equation (D.E.) is any equation connecting an independent variable x, dependent variable y and it's derivative $y', y'', y''' \dots y^{(n)}$ i.e. the equation of form $F(x, y, y', y'', \dots, y^{(n)}) = 0$

For example:

1. $y'' + 3y = x^2$
2. $4y'' + xy' + x^2 y = \sin x$
3. $yy'' + xy' + 6y = \cosh x$

There are two kinds of D.E.

1. Ordinary Differential Equations: المعادلات التفاضلية الاعتيادية
   which contain one independent variable, for exaple
   $$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 7y = e^{x}$$

2. Partial Differential Equations: المعادلات التفاضلية الجزئية
   which contain two or more independent variables

   For example:

   1. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$
   2. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} + \dfrac{\partial^2 z}{\partial x \partial y} = 0$

Definition (2): The رتبة Order of D.E. is the order of the highest derivative contained in the equation.

Definition (3): The درجة degree of D.E. is the power of highest derivative in the equation.

For example,

1. $\frac{dy}{dx} = x^2 + 3$, first order , first degree
2. $x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + y = x^2$, second order, first degree
3. $\left(\frac{d^3y}{dx^3}\right)^2 + 2\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$, third order, second degree
4. $\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^2 + 5\frac{dy}{dx} = e^x$, third order , first degree
5. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 10$, second order, first degree
6. $(y'')^{1/3} - (x-y)^{1/2} = 0$, second order, second degree

H.w. Determine the order and the degree of the following D.E.

1. $y''' - 3xy' + y = e^x + 1$
2. $xy'' + xy' + \cos\sqrt{y} = x^2 + 1$
3. $2(y')^2 - (y')^9 + y^8 - x = 0$
4. $(y'')^2 - xy\sin x + y' = 0$

Definition (4): Any Differential Equation with any order will be
خطية
linear if the function and it's derivatives of first degree ad not multiply one another. For example:

1. $xy'' + xy' + 5y = 0$ Linear D.E.
2. $(y'')^2 + 6y' + 5y = 10$ non-linear D.E.

Definition (5): The general form of linear Differential
معاملات ثابتة
equation with constant coefficients is :

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = f(x)$$

where $a_0 \neq 0$, $a_0, a_1, \ldots, a_n$ are constants

متجانسة
1. If $f(x) = 0$, then D.E. called homogenous
غير متجانسة
2. If $f(x) \neq 0$, then D.E. called non-homogenous

حل المعادلات التفاضلية
Definition (6): The solution of Differential equation is any relation between variables of D.E. such that this relation :

1. without derivative
2. Defined on the interval
3. satisfying the D.E.

Kinds of Solution:

الحل العام

1. General Solution: this solution is satisfying D.E. and contain arbitrary constant.

الحل الخاص

2. Particular Solution: we take this solution from general solution when we give values of the arbitrary constant.

الحل المنفرد

3. Singular Solution: A solution which can not be obtained from a general solution by putting any value of the arbitrary constant.

Example (1): The function $y=e^{2x}$ is a solution of the Differential Equation $y''-3y'+2y=0$

Solution: The function $y=e^{2x}$ has no derivative and defined on the interval $(-\infty,\infty)$ also $y'=2e^{2x}$,

$y''=4e^{2x}$ defined on the same interval and

$$y''-3y'+2y=4e^{2x}-6e^{2x}+2e^{2x}=0.$$

so it is a solution of the equation.

Example (2): show that the function $\ln y=Ae^{x}+Be^{-x}$, $y>0$ is a solution of $yy''-(y')^2=y^2\ln y$. where A,B constant

Solution: $\ln y = Ae^{x} + Be^{-x}$

نشتق ضمنياً الطرفين

$$\Rightarrow \frac{1}{y}y' = Ae^{x} - Be^{-x}$$

$$\Rightarrow y^{-1}y'' - y^{-2}(y')^2 = Ae^{x} + Be^{-x}$$

$$\Rightarrow y^{-1}y'' - y^{-2}(y')^2 = \ln y$$

بضرب الطرفين في y^2

$$\Rightarrow yy'' - (y')^2 = y^2 \ln y$$

نحصل على المعادلة

we get $\ln y = Ae^{x} + Be^{-x}$, $y>0$ is a solution of D.E.

Example (3): Show that $(y-A)^2 = Ax$ is a general solution of the differential equation $4x(y')^2 + 2xy' - y = 0$, then find the particular solution when $y(1) = 2$.

Solution: Since $(y-A)^2 = Ax$, so $y' = \dfrac{A}{2(y-A)}$ بالاشتقاق الضمني

Now $$4x(y')^2 + 2xy' - y = 4x\frac{A^2}{4(y-A)^2} + 2x\frac{A}{2(y-A)} - y$$

$$= \frac{xA^2 + xA(y-A) - y(y-A)^2}{(y-A)^2}$$

$$= \frac{xAy - y(y-A)^2}{(y-A)^2}$$

بالتعويض عن Ax

$$= \frac{y(y-A)^2 - y(y-A)^2}{(y-A)^2} = \underline{\underline{0}}$$

This mean that $(y-A)^2 = Ax$ is the general solution

To find the particular solution we use $y = 2$, $x = 1$

in $(y-A)^2 = Ax \Rightarrow (2-A)^2 = A \Rightarrow A^2 - 5A + 4 = 0$

$\Rightarrow A = 4$ or $A = 1$ so $(y-4)^2 = 4x$ or $(y-1)^2 = x$ is the particular solution

Formulation of D.Q. from the general solution

Here, we study the problem of finding D.E. if we know the general solution, we note the relation between the numbers of arbitrary constants and the order of D.E.

Example (4): Find the D.E. which has general solution of the form $y = c_1 x + c_2 x^3$ --- (1)

Solution: we have two arbitrary constants, hence we differentating twice لايجاد المعادلة نشتق الحل العام مرتين لاحتوائه على ثابتين

$y' = c_1 + 3c_2 x^2$, $y'' = 6c_2 x$

from these equations, we find $c_2 = \frac{y''}{6x}$, $x \neq 0$

substituting in first equation

$$y' = c_1 + 3\left(\frac{y''}{6x}\right)x^2 \Rightarrow c_1 = y' - \frac{1}{2}xy''$$

substituting c_1, c_2 in (1) we get

$$y = \left(y' - \frac{1}{2}xy''\right)x + \frac{y''}{6x}\cdot x^2$$

$\therefore$ $y = xy' - \frac{1}{3}x^2 y''$ be the required differential eq.

substituting (1) in the equation. التحقيق

$$x(c_1 + 3c_2 x^2) - \frac{1}{3}x^2(6c_2 x) = c_1 x + c_2 x^3$$

$$\Rightarrow c_1 x + c_2 x^3 = c_1 x + c_2 x^3$$

L.H.S = R.H.S.

__Example (5):__ Find the D.E. which has solution of the form $y = Ax + Bx^4$ and verify the result
نتحقق من النتيجة

__Solution:__ we have two arbitrary constants, hence we differentating twice نشتق مرتين

$y' = A + 4Bx^3$, $y'' = 12Bx^2 \Rightarrow B = \frac{y''}{12x^2}$ from this we find A

$$y' = A + 4\left(\frac{y''}{12x^2}\right)x^2 \Rightarrow y' = A + \frac{y''}{3}x \Rightarrow A = y' - \frac{y''}{3}x$$

substituting in the equation $y = A + Bx^4$, we get

$$y = \left(y' - \frac{y''}{3}x\right)x + \frac{y}{12x^2}x^4$$

$$\Rightarrow x^2y'' - 4xy' + 4y = 0$$ which is the required D.E.

__التحقيق__: نعوض في المعادلة الاصلية بعد الاشتقاق نجد

$$x^2\left(\frac{12x^2y''}{12x^2}\right) - 4x\left(y' - \frac{y''}{3}x + \frac{y''}{3}x\right) + 4\left[\left(y' - \frac{y''}{3}x\right)x + \frac{y''}{12x^2}\cdot x^4\right] = 0$$

$\therefore$ L.H.S = R.H.S.

__Example (6):__ Find the D.E. which has the solution of the form $y = Ae^x + Be^{2x}$, where A, B arbitrary constant

Solution: we differentating $y = Ae^{x} + Be^{2x}$ twice

we get $y' = Ae^{x} + 2Be^{2x}$

$y'' = Ae^{x} + 4Be^{2x}$

وحل المعادلات آنياً

$$y' - y = Be^{2x}$$

$$\Rightarrow \quad y'' - y' = 2Be^{2x} \Rightarrow y'' - y' = 2(y' - y)$$

$$\therefore \quad y'' - y' - 2y' + 2y = 0 \Rightarrow y'' - 3y' + 2y = 0$$ the required eq.

Exercises

(1.) Determine the Order and the degree for each of the following D.E.

1. $y''' + xy'' + 2y(y')^2 + xy = 0$
2. $e^{y'''} - xy'' + y = 0$
3. $2(y'')^2 - (y')^9 + y^3 - x = 0$
4. $\frac{d^2v}{dx^2}\frac{dv}{dx} + x\left(\frac{dv}{dx}\right)^2 + v = 0$
5. $\sqrt{P' + P} = \sin\theta$

(2.) In each of the following, show that the given functions are a general solution of the D.E.

1. $y = \frac{1}{x}e^{Ax}, x \neq 0 \qquad xy' + y - y\ln x = 0$
2. $y = \tan^{-1}(x + A) \quad , \quad y' - \cos^2 y = 0$
3. $y = Ax + Bx\ln x + \frac{1}{2}x(\ln x)^2 , \quad x^2y'' - xy' + y = x(\ln x)^3$

(3) Find the values of constant c which implies $y = e^{cx}$ is solution of D.E. $y'' + 5y' + 6y = 0$

(4) Find the value of constant A which implies $y = Ax + 2$ is solution of D.E. $y'' + 5y' + y = 5$

(5) Find the differential equation which has the general solution given in each of the following

1. $\ln y = Ax^2 + B$, A, B constant ans: $xyy'' - yy' + x(y')^2 = 0$

2. $y^2 = Ax + B$, " ans: $yy'' + (y')^2 = 0$

3. $y = A\sin x$ " ans: $y' = y\cot x$

4. $y = Ae^x + B$ " ans: $y'' - y' = 0$

مسألة القيمة الابتدائية

<u>Definition (7):</u> The initial value problem (Cauchy problem) (I.V.P.) is a problem of finding the solution $y = y(x)$ of the equation $y' = f(x, y)$, satisfying the initial conditions $y(x_0) = y_0$

For example:

The D.E. $\frac{dy}{dx} + y = 0$ with $y(1) = 3$ is I.V.P.