Example (4): Given $S = \{(x,y) : |x| \leq 1 , |y| \leq 1\}$

and $y' = x^2 + y^2$, $y(0) = 0$ --- ①

1- Is the I.v.p. (1) satisfy the conditions of Cauchy-Peano theorems?

2- use picard method to find the g_1, g_2, g_3 of I.v.p. (1)

Solution: ① Since $f(x,y) = x^2 + y^2$ is Polynomail, then it is continuous in R^2, i.e. it is Continuous in S

② $|f(x,y)| = |x^2 + y^2| \leq |x^2| + |y^2| \leq 1 + 1 \leq 2$

So, $M = 2$ and the function bounded in S

③ $I_h : |x - x_0| \leq h$, $h = \min(a, \frac{b}{M}) = \min(1, \frac{1}{2}) = \frac{1}{2}$

$\therefore I_h : |x| \leq \frac{1}{2}$ in this interval there exist a solution

④ Since $f(x,y) = x^2 + y^2$, $\frac{\partial f}{\partial y} = 2y$ continuous in S

So the function satisfies lipschitz Condition in S

i.e. the I.v.p. has a unique solution

To find the solution by Picard approximation

$$g_n(x) = y_0 + \int_0^x f[t, g_{n-1}(t)]\, dt \qquad n = 1, 2, 3, \ldots$$

$$g_0(x) = y_0 = 0$$

$$g_1(x) = \int_0^x f(t, g_0(t))\,dt = \int_0^x t^2\,dt = \frac{x^3}{3}$$

$$g_2(x) = \int_0^x f(t, g_1(t))\,dt = \int_0^x \left[t^2 + \left(\frac{t^3}{3}\right)^2\right] dt$$

$$= \int_0^x \left[t^2 + \frac{t^6}{9}\right] dt = \frac{x^3}{3} + \frac{x^7}{63}$$

$$g_3(x) = \int_0^x f(t, g_2(t))\,dt = \int_0^x \left[t^2 + \left(\frac{t^3}{3} + \frac{t^7}{63}\right)^2\right] dt$$

$$= \int_0^x \left[t^2 + \frac{t^6}{9} + \frac{2t^{10}}{189} + \frac{t^{14}}{3969}\right] dt$$

$$= \frac{x^3}{3} + \frac{x^7}{63} + \frac{2t^{11}}{189.11} + \frac{t^{15}}{3969.15}$$

Example (5): Let $y' = xy$, $-\infty < x < \infty$, $-\infty < y < \infty$ with $y(0) = 1$, use Picard method to find the solution of the I.V.P.

Solution: Since $f(x,y) = xy$ continous on R^2 and $\frac{\partial f}{\partial y} = x$ also continuous, thus the problem have a unique solution

Let $\{g_n\}$ be a sequences defined on R s.t.

$$g_0(x) = 1$$

$$g_n(x) = 1 + \int_0^x f(t, g_{n-1}(t))\,dt \quad , n \geq 1 \quad , -\infty < x < \infty$$

for $n=1$

$$g_1(x) = 1+\int_0^x f(t, g_0(t))\,dt = 1+\int_0^x t\,dt = 1+\frac{x^2}{2} = 1+\frac{x^2}{2!}$$

$$n=2 \quad g_2(x) = 1+\int_0^x f(t, g_1(t))\,dt = 1+\int_0^x t\,g_1(t)\,dt$$

$$= 1+\int_0^x t\left(1+\frac{t^2}{2}\right)dt = 1+\frac{x^2}{2!}+\frac{\left(\frac{x^2}{2}\right)^3}{3!}$$

and so on

then $g_n(x) = 1+\sum_{i=1}^{n} \frac{\left(\frac{x^2}{2}\right)^i}{i!} \quad \dots *$

take lim to both sides of $*$ we get

$$g(x) = \lim_{n\to\infty} 1+\sum_{i=1}^{\infty} \frac{\left(\frac{x^2}{2}\right)^i}{i!} = e^{x^2/2}$$

Exercises

1. In each of the following, prove that the function f is satisfying Lipschitz condition in the given rejon.

a. $f(x,y) = \frac{\cos y}{1-x^2}$, $S=\{|x|<1 \ , \ |y|<\infty\}$

b. $f(x,y) = x^3 e^{-x} y^2$, $S=\{0\leq x\leq a \ , \ |y|<\infty\}$

c. $f(x,y) = ax^2+by^2$ $a,b\in R$, $S=\{|x|\leq 1, \ |y|\leq 1\}$

2. Find the value of Lipschitz Constant for each of the following

a. $f(x,y) = 4x^2+y^2$, $S=\{ |y-1|\leq 2 , |x|\leq 1\}$

b. $f(x,y) = x^2\cos^2 y + y\sin^2 x$ $S=\{(x,y): |x|\leq 1 , |y|<\infty\}$

3. Find g_0, g_1, g_2, g_3 of Picard approximation to each of the following I.V.P

a. $y' = y^2$, $y(0)=1$

b. $y' = x^2+y^2$, $y(0)=1$

4. Let $f(x,y) = 3y+1$, (x,y) in $\mathbb{R}^2$, $y(0)=2$

a. Does the function f satisfy Cauchy-peano theorems?

b. Find the interval for which the solution defined

c. Find $g_0, g_1, \ldots, g_n$ of picard approximation

5. Solve the initial value Problem $\frac{dy}{dx} = -y-1$ with $y(0)=0$ using the method of Picard (Successive appr.)

Linear Differential Equations of n-th order

المعادلات التفاضلية الخطية من الرتبة n

In general these D.E. has the form:

$$a_0(x)\frac{d^n y}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_n(x)y=h(x) \quad \text{---} ①$$

where $a_0(x), a_1(x), \dots, a_n(x)$ are Continuous functions such that $a_0(x)\neq 0$

If $a_0, a_1, a_2, \dots, a_n$ constants, then D.E. Called D.E with constant coefficients and has the form:

$$a_0\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_n y=h(x) \quad \text{---} ②$$

$a_0\neq 0$

If $h(x)=0$ then D.E. called homogenous متجانسة

If $h(x)\neq 0$ then D.E. called nonhomogenous غير متجانسة

If L_n is operator مؤثر defined by the following

$$L_n=a_0(x)\frac{d^n}{dx^n}+a_1(x)\frac{d^{n-1}}{dx^{n-1}}+\cdots+a_n(x)$$

then we can rewrite D.E. ① as following

$$L_n y=h(x)$$

Linear Differential Equations of n-th order with Variable coefficients المعادلات التفاضلية الخطية من الرتبة n ذات المعاملات المتغيرة

Theorem (3): D.E. (1) equivalent تكافئ the following D. system

$$y_1 = y$$

$$y_1' = y_2 = y'$$

$$y_2' = y_3 = y''$$

$$\vdots$$

$$y_{n-1}' = y_n = y^{(n-1)}$$

$$y_n' = -\frac{1}{a_0(x)}\left(a_1(x)y_n + a_2(x)y_{n-1} + \cdots + a_n(x)y_1\right) + \frac{h(x)}{a_0(x)}$$

Example (6): Transform حول the I.V.P. $y'' + 2yy' - t^2y = 0$, $y(0) = 0$, $y'(0) = 1$, to D. system of first order

Solution: Let $y_1 = y$

$$y_2 = y'$$

$$\Rightarrow \quad y_1' = y' = y_2$$

$$y_2' = y'' = t^2y - 2yy'$$

i.e.

$$y_1' = y_2$$

$$y_2' = t^2y - 2y_1y_2$$

with $y_1(0) = 0$, $y_2(0) = 1$