Differential Equations of first order and first degree

المعادلات التفاضلية من الرتبة الاولى والدرجة الاولى

This D.E. has the following form

$$g(x,y) + h(x,y)\,\frac{dy}{dx} = 0$$

or

$$g(x,y)\,dx + h(x,y)\,dy = 0$$

or

$$\frac{dy}{dx} = f(x,y)$$

where $g(x,y)$, $h(x,y)$ and $f(x,y)$ are functions of x and y

This D.E. can solving directly and in the following, the kinds of this D.E.

1. Equations of separation variables المعادلات التي تنفصل متغيراتها
2. Homogenous equations المعادلات المتجانسة
3. Differential equations with linear Coefficients المعادلات التفاضلية ذات المعاملات الخطية
4. Exact and non-exact equations المعادلات التفاضلية التامة وغير التامة
5. linear D.E. and Bernoli equation المعادلات الخطية ومعادلة برنولي

1. Equations of separation variables.

This D.Es are of simple types of D.E. which can writting as

$$g(x)\,dx + h(y)\,dy = 0$$

and the general solution will be

$$\int g(x)\,dx + \int h(x)\,dx = c \quad , \ c \text{ is arbitrary Con.}$$

Example (7): Solve the D.E. $y\,dx - x\,dy = 0$

Solution: dividing by xy we get

$$\frac{dx}{x} - \frac{dy}{y} = 0 \xrightarrow{\text{تكامل}} \ln|x| - \ln|y| = \ln|c|$$

$$\ln\left|\frac{x}{y}\right| = \ln c \Rightarrow \frac{x}{y} = c \Rightarrow x = cy \text{ is the Sol.}$$

Example (8): Solve the D.E. $(\sin x + \cos x)\,dy + (\cos x - \sin x)\,dx = 0$

Solution: dividing by $\sin x + \cos x$, we get

$$dy + \frac{\cos x - \sin x}{\sin x + \cos x}\,dx = 0$$

integrating both sides, we get

$$y + \ln|\sin x + \cos x| = c \quad , \text{ the solution of D.E.}$$

Example (9): Solve the D.E. $x^2(1-y^2)\,dx + y(1+x^2)\,dy = 0$

solution: dividing by $(1-y^2)(1+x^2)$ we get

$$\frac{x^2}{1+x^2}\,dx + \frac{y}{1-y^2}\,dy = 0$$

$$\begin{array}{r|l} & 1 \\ x^2+1 & x^2 \\ & \mp x^2 \mp 1 \\ \hline & -1 \end{array}$$

$$\left(1 - \frac{1}{1+x^2}\right)dx + \frac{y}{1-y^2}\,dy = 0$$

تكامل

$$x - \tan^{-1}x + \frac{1}{2}\ln|1-y^2| = c$$

Example (10): Solve the D. E.

$$xy\,dy + (2yx^2 + 4x^2 - y - 2)\,dx = 0$$

Solution:

$$xy\,dy + \left[2x^2(y+2) - (y+2)\right]dx = 0$$ ناخذ عامل مشترك

$$xy\,dy + (y+2)(2x^2-1)\,dx = 0$$ معادلة تفاضلية قابلة للفصل

dividing by $x(y+2)$, we get

$$\frac{y}{y+2}\,dy + \frac{2x^2-1}{x}\,dx = 0$$

$$\left(1 - \frac{2}{y+2}\right)dy + \left(2x - \frac{1}{x}\right)dx = 0$$

$$\begin{array}{r|l} & 2x \\ \hline x & 2x^2-1 \\ & \mp 2x^2 \\ \hline & -1 \end{array} \qquad \begin{array}{r|l} & 1 \\ \hline y+2 & y \\ & \mp y \mp 2 \\ \hline & -2 \end{array}$$

integrating both sides

$$y - 2\ln|y+2| + x^2 - \ln|x| = c$$ is the sol. of D.E.

Example (11): Find the particular solution for the D.E. (الحل الخاص)

$$2x(y+1)\,dx - y\,dy = 0$$ that satisfy $y(0) = -2$

Solution; dividing by $(y+1)$, we get

$$2x\,dx - \frac{y}{y+1}\,dy = 0$$

$$\begin{array}{r|l} & 1 \\ \hline y+1 & y \\ & \mp y \mp 1 \\ \hline & -1 \end{array}$$

integrating, we have

$$x^2 - y + \ln|y+1| = c$$ which is the general sol..

to find the particular soln. we subsititution $x=0$, $y=-2$

we get $0 + 2 + \ln|-2+1| = c \quad \therefore c = 2$

So the solution is $x^2 - y + \ln(y+2) = 2$.

Exercises

① Solve the following D.Es.

1. $(y^2+y)\,dx-(x^2-x)\,dy=0$
2. $x^2(1-y)\,dx+y(1+x^2)\,dy=0$
3. $\sin^2 x\cos y\,dx+\sin y\sec x\,dy=0$
4. $x(1-y)\dfrac{dy}{dx}+y(1+x)=0$
5. $y\,dx-(e^{2x}+4)\,dy=0$

② Find the particular solution of the following D.E.

1. $(1+e^{-y})\sin x\,dx=(1+\cos x)\,dy\ ,\quad y(0)=0$
2. $dy=2(2y\,dx-x\,dy)\ ,\quad y(1)=4.$
3. $(y+2)\,dx+y(x+4)\,dy=0 \quad y(-3)=-1$

2. Homogenous differential equations المعادلات التفاضلية المتجانسة

Definition (8): A function $f(x,y)$ is said to be homogenous (متجانسة) function (دالة) of it's variable of degree n (من الدرجة n) if the identity

$$f(tx,ty)=t^n f(x,y) \text{ is satisfying } \forall\ x,y,t\in\mathbb{R}$$

Example (12): The function $f(x,y)=7x^2+2xy+5y^2$ homogenous of second degree since.

$$f(tx,ty)=7(tx)^2+2(tx)(ty)+5(ty)^2$$
$$=7t^2x^2+2t^2xy+5t^2y^2$$

$$= t^2[7x^2 + 2xy + 5y^2] = t^2 f(x,y)$$

Example(13): The function $f(x,y) = x+y$ homogenous of first degree since:

$$f(tx, ty) = tx + ty = t[x+y] = t f(x,y)$$

Definition(9): The D.E. $M(x,y)\,dx + N(x,y)\,dy = 0$ is called homogenous if $M(x,y)$, $N(x,y)$ are homogenous functions and of the متشابه الدرجة same degree

Example(14): Solve the D.E. $(xy - y^2)\,dx - x^2\,dy = 0$ ...(1)

solution: نختبر اولاً خاصية التجانس لكل من M و N

$$M(x,y) = xy - y^2, \quad M(xt, ty) = (tx)(ty) - (ty)^2$$
$$= t^2[xy - y^2] = t^2 M(x,y)$$

$\therefore$ $M(x,y)$ homogenous of second degree.

$$N(x,y) = x^2, \quad N(tx, ty) = (tx)^2 = t^2 N(x,y)$$

$\therefore$ $N(x,y)$ homogenous of second degree.

so $M(x,y)$, $N(x,y)$ homogenous of نفس same degree

$\therefore$ The D.E (1) homogenous

To find the solution of homogenous D.E. first we

Let $y = vx$, $dy = v\,dx + x\,dv$

substituting in (1)

$$(xvx - v^2x^2)dx - x^2(v\,dx + x\,dv) = 0$$

$$(x^2v - v^2x^2)dx - x^2(v\,dx + x\,dv) = 0$$

$$x^2(v - v^2)dx - x^2(v\,dx + x\,dv) = 0$$

divide by x^2

$$(v - v^2)dx - (v\,dx + x\,dv) = 0$$

$$v(1 - v)dx - v\,dx - x\,dv = 0$$

$$v\,dx - v^2dx - v\,dx - x\,dv = 0$$

$$-v^2dx - x\,dv = 0 \qquad \text{multiply by } (-1)$$

$v^2dx + x\,dv = 0$ is a D.E. of separation variable قابلة للفصل

dividing by xv^2, we get

$$\frac{dx}{x} + \frac{dv}{v^2} = 0$$

integrating

$$\ln|x| + \frac{v^{-1}}{-1} = \ln c$$

$$\ln x - \frac{1}{v} = \ln c$$

نعوض عن $y = vx \Leftarrow v = \frac{y}{x}$ في الحل العام

$$\ln x - \frac{1}{y/x} = \ln c \Rightarrow \ln x - \frac{x}{y} = \ln c \Rightarrow \ln x - \ln c = \frac{x}{y}$$

بالتبسيط

$$\Rightarrow \ln\frac{x}{c} = \frac{x}{y} \Rightarrow y = \frac{x}{\ln x/c}$$

ملاحظة: خطوات ايجاد حل للمعادلة المتجانسة

1. نفرض $y = vx$ ونضع $dy = v\,dx + x\,dv$
2. نعوض عن y و dy في المعادلة وبعد التبسيط تتحول الى قابلة للفصل ونجد الحل
3. نعوض عن $v = \frac{y}{x}$ في الحل الناتج وبالتبسيط نحصل على الحل للمعادلة الاصلية

Example (15): Solve the D.E. $(2x-y)dy = (2y-x)dx$

Solution: $M(x,y) = 2y-x$, $M(tx,ty) = 2ty - tx = tM(x,y)$

$N(x,y) = 2x-y$, $N(tx,ty) = 2tx - ty = tN(x,y)$

$\therefore$ M, N homogenous of first degree.

$\therefore$ D.E. is homogenous.

Let $y = vx$, $dy = v\,dx + x\,dv$

substituting in the eq.

$$(2vx - x)dx - (2x - vx)(v\,dx + x\,dv) = 0$$

$$2vx\,dx - x\,dx = 2xv\,dx + 2x^2dv - v^2x\,dx - vx^2dv$$

$$(v^2x - x)dx + (vx^2 - 2x^2)dv = 0$$

$$x(v^2-1)dx + x^2(v-2)dv = 0$$ D.E. of separation variable قابلة للفصل

$$\frac{dx}{x} + \frac{v-2}{v^2-1}dv = 0$$ divide by $x^2(v^2-1)$

Integrating

نلاحظ ان الكسر الثاني المقام فيه اعلى من البسط لهذا نستخدم طريقة الكسور الجزئية في التكامل موضح تحت

$$\frac{dx}{x} + \left[\frac{3/2}{v+1} + \frac{-1/2}{v-1}\right]dv = 0$$

$$\ln x + \frac{3}{2}\ln|v+1| - \frac{1}{2}\ln|v-1| = c$$

نرجع $v = \frac{y}{x}$

$$\Rightarrow \ln x + \frac{3}{2}\ln\left|\frac{y}{x}+1\right| - \frac{1}{2}\ln\left|\frac{y}{x}-1\right| = c$$

الحل العام

$$\frac{v-2}{v^2-1} = \frac{v-2}{(v-1)(v+1)} = \frac{A}{v+1} + \frac{B}{v-1} = \frac{A(v-1)+B(v+1)}{(v+1)(v-1)}$$, we find $\Rightarrow$ $A = \frac{3}{2}$, $B = -1/2$

Example (16): Solve the D.E. $x\,dy - y\,dx - \sqrt{x^2 - y^2}\,dx = 0$

Solution: This D.E. is homogenous of degree 1 (H.W.)

Let $y = vx$, $dy = v\,dx + x\,dv$

نعوض في المعادلة الاصلية

$$x[v\,dx + x\,dv] - vx\,dx - \sqrt{x^2 - v^2x^2}\,dx = 0$$

$$vx\,dx + x^2\,dv - vx\,dx - x\sqrt{1 - v^2}\,dx = 0$$

قابلة للفصل

$\Rightarrow \; x^2\,dv - x\sqrt{1 - v^2}\,dx = 0$ is D.E of separation variable

$$\frac{dv}{\sqrt{1 - v^2}} - \frac{dx}{x} = 0 \qquad \text{divide by } x^2\sqrt{1 - v^2}$$

integrating, we get

$$\sin^{-1} v - \ln x = \ln c$$

$$\sin^{-1} v = \ln c + \ln x$$

نعوض عن $v = \frac{y}{x}$ في كل مكان

$$\sin^{-1} \frac{y}{x} = \ln c + \ln x$$

$$\therefore \; \sin^{-1} \frac{y}{x} = \ln xc$$

$\therefore \; xc = e^{\sin^{-1} \frac{y}{x}}$ is the solution of D.E.

Example (17): Find the solution of the D.E.

$$x\,dx + \sin^2\left(\frac{y}{x}\right)[y\,dx - x\,dy] = 0$$

Solution: This D.E. is homogenous of degree 1 (H.W.)

Now Let $y = vx \Rightarrow dy = v\,dx + x\,dv$

نعوض في المعادلة

$$x\,dx + \sin^2(v)\left[v x\,dx - x(v\,dx + x\,dv)\right] = 0$$

بتبسيط المعادلة نحصل على

$$x\,dx + x^2 \sin^2(v)\,dv = 0$$

divide by x^2

$$\frac{dx}{x} - \sin^2(v)\,dv = 0$$

$$\Rightarrow \frac{dx}{x} - \frac{1-\cos(2v)}{2}\,dv = 0$$

integrating we get

$$\ln|x| - \frac{1}{2}v + \frac{1}{4}\sin 2v = c$$

نعوض عن $v = \frac{y}{x}$ بالحل الناتج

$$\ln|x| - \frac{1}{2}\frac{y}{x} + \frac{1}{4}\sin\frac{2y}{x} = c$$

multiply by $4x$

So $4x\ln|x| - 2y + x\sin\frac{2y}{x} = 4cx$ is the solution.

Exercises

Solve each of the following D.Es.

1. $(x^2+y^2)\,dx - 2xy\,dy = 0$

2. $xy^2\,dy - (x^3+y^3)\,dx = 0$

3. $(2xy+y^2)\,dx - 2x^2\,dy = 0$

4. $(xy+y^2)\,dx + x^2\,dy = 0$

5. $\dfrac{dx}{dy} = \dfrac{xy^3}{2y^4+x^4}$

6. $\dfrac{dy}{dx} = \dfrac{x^2+y^2}{xy}$, $y(1) = -2$