Example (7): Transform the I.v.p. $y'' + y = 0$ with $y(0) = 0$, $y'(0) = 1$, to the D. system of first order

Solution: Let $y_1 = y$, $y_2 = y'$

$$\Rightarrow \quad y_1' = y' = y_2$$

$$y_2' = y'' = -y$$

i.e. we get

$$\left.\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_1 \end{aligned}\right\} \text{ D. system with } y_1(0) = 0,\ y_2(0) = 1$$

Theorem (4): If the functions $a_0, a_1, a_2, \dots, a_n$ continuous functions on I, such that $a_0(x) \neq 0$, $\forall x \in I$, then the D.E. (1) has unique (وحيد) solution on I.

Definition (5): Given the functions $g_1, g_2, \dots, g_n$ and the constants $c_1, c_2, \dots, c_n$ not all zero exist such that:

$$c_1 g_1(x) + c_2 g_2(x) + \dots + c_n g_n(x) = 0 \quad \text{---} \quad (1)$$

, then $g_1(x), g_2(x), \dots, g_n(x)$ called linearly dependent (خطياً مرتبطة) functions, identically in some interval $a \leq x \leq b$, $g_1(x), g_2(x), \dots, g_n(x)$ is said to be linearly independent (مستقلة خطياً) functions if eq. (1) implies

$$c_1 = c_2 = \dots = c_n = 0$$

Example (8): show that the functions e^{x}, e^{2x} are linearly independent.

Solution: from eq. (1), we have $c_1y_1 + c_2y_2 = 0$, then

$$c_1e^{x} + c_2e^{2x} = 0$$

بالطرح
$$\mp c_1e^{x} \mp 2c_2e^{2x} = 0$$
$$\overline{\quad -c_2e^{2x} = 0 \quad}$$

Since $e^{2x} \neq 0$, then $c_2 = 0$

$$c_1e^{x} = 0 \quad \text{نعوضها في المعادلة الأولى فنجد}$$

Since $e^{x} \neq 0$, then $c_1 = 0$

So $c_1 = c_2 = 0$, then the functions are linearly independent

Example (9): show that the functions $x^2 - x, 2x^2, 3x$ are linearly dependent

Solution: Let $c_1y_1 + c_2y_2 + c_3y_3 = 0$, then

$$c_1(x^2 - x) + 2c_2x^2 + 3c_2x = 0$$

$$c_1(2x - 1) + 4c_2x + 3c_2 = 0 \qquad \text{نشتق}$$

$$2c_1 + 4c_2 = 0 \qquad \text{نشتق مرة ثانية}$$

$$\Rightarrow c_1 = -2c_2$$

نعوضها في المعادلتين (2), (3)

$$\Rightarrow 2c_2x + 8c_2x = 0$$

$$2c_2 + 3c_3 = 0$$

$$\Rightarrow c_2 = -\frac{3}{2}c_3, \quad c_1 = 3c_3$$

ولهذه المعادلات أقل من عدد المجاهيل وبهذه الحالة لايمكن ان يوجد حل وحيد ، نفرض $c_3 = 2$ فنحصل $c_2 = -3$ ، $c_1 = 6$ وعليه الدوال معتمدة خطياً

Theorem (5): The D.E. $L_n y = 0$ has n of linearly independent solutions on the interval I.

Definition (6): Given the functions $g_1, g_2, \ldots, g_n$ and

$$W(g_1, g_2, \ldots, g_n)(x) = \begin{vmatrix} g_1(x) & g_2(x) & \cdots & g_n(x) \\ g_1'(x) & g_2'(x) & \cdots & g_n'(x) \\ \vdots & \vdots & & \\ g_1^{(n-1)}(x) & g_2^{(n-1)}(x) & \cdots & g_n^{(n-1)}(x) \end{vmatrix}$$

$\forall x \in I$, then $W(g_1, g_2, \ldots, g_n)(x)$ is called wronsikan

Example (10): Given x, $\sin x$, then

$$W(x, \sin x) = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x\cos x - \sin x$$

Example (11): Given e^{x}, e^{-x}, e^{2x}, then

$$W(e^{x}, e^{-x}, e^{2x}) = \begin{vmatrix} e^{x} & e^{-x} & e^{2x} \\ e^{x} & -e^{-x} & 2e^{2x} \\ e^{x} & e^{-x} & 4e^{2x} \end{vmatrix}$$

$$= \left|\begin{matrix} e^{x} & e^{-x} & e^{2x} \\ e^{x} & -e^{-x} & 2e^{2x} \\ e^{x} & e^{-x} & 4e^{2x} \end{matrix}\right| \begin{matrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \\ e^{x} & e^{-x} \end{matrix} = -6e^{2x}$$

Theorem (6): If $g_1, g_2, \cdots, g_n$ are solutions of the D.E. $L_n y = 0$ in the interval I, then $g_1, g_2, \cdots, g_n$ linearly independent iff $\forall x \in I$, then $w(g_1, g_2, \cdots, g_n)(x) \neq 0$.

Proof:

Let $w(g_1, g_2, \cdots, g_n) \neq 0$, and $c_1, c_2, \ldots, c_n$ real numbers such that $\forall x \in I$

$$c_1 g_1(x) + c_2 g_2(x) + \cdots + c_n g_n(x) = 0 \quad \cdots (1)$$

Since $g_1, g_2, \cdots, g_n$ are solutions of $L_n y = 0$ on I and g_i have derivative of the (n-th) order, then

$$\left.\begin{aligned} c_1 g_1'(x) + c_2 g_2'(x) + \cdots + c_n g_n'(x) &= 0 \\ c_1 g_1''(x) + c_2 g_2''(x) + \cdots + c_n g_n''(x) &= 0 \\ &\vdots \\ c_1 g_1^{(n-1)}(x) + c_2 g_2^{(n-1)}(x) + \cdots + c_n g_n^{(n-1)}(x) &= 0 \end{aligned}\right\} \cdots (2)$$

by using Grammer rule to solve (1) and (2), we get

$$c_1 = \frac{\begin{vmatrix} 0 & g_2 & \cdots & g_n \\ 0 & g_2' & \cdots & g_n' \\ \vdots & \vdots & \vdots & \vdots \\ 0 & g_2^{(n-1)} & & g_n^{(n-1)} \end{vmatrix}}{\begin{vmatrix} g_1 & g_2 & \cdots & g_n \\ g_1' & g_2' & \cdots & g_n' \\ \vdots & \vdots & \cdots & \vdots \\ g_1^{(n-1)} & g_2^{(n-1)} & & g_n^{(n-1)} \end{vmatrix}} \Rightarrow c_1 = \frac{0}{w(g_1, g_2, \cdots, g_n)}$$

$\Rightarrow c_1 W(g_1, g_2, \dots, g_n)(x) = 0$

Similarly, $c_2 W(g_1, g_2, \dots, g_n)(x) = 0$, hence

$c_i W(g_1, g_2, \dots, g_n)(x) = 0, \quad i = 1, 2, \dots, n$

Since $W(g_1, g_2, \dots, g_n)(x) \neq 0$, then $c_i = 0$

$\therefore g_1, g_2, \dots, g_n$ linearly independent

<u>Conversly</u> العكس

Let $g_1, g_2, \dots, g_n$ linearly independent solutions and Let $W(g_1, g_2, \dots, g_n)(x_0) = 0$ for any point x_0 in I.
and Let $c_1, c_2, \dots, c_n$ real numbers such that $\forall x \in I$

$c_1 g_1(x) + c_2 g_2(x) + \dots + c_n g_n(x) = 0$

in the same method we get

$c_i W(g_1, g_2, \dots, g_n)(x) = 0$, $i = 1, 2, \dots, n$, unnecessary (ليس بالضرورة)
$c_i = 0$ and this contradiction (تناقض) our assumption (linearly independent), hence $W(g_1, g_2, \dots, g_n)(x) \neq 0 \ \forall x \in I$.

<u>Example (12)</u>: show that $\cos x$, $\sin x$ are linearly independent solutions from wronskian

$$W(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0$$

$\therefore y_1 = \cos x$, $y_2 = \sin x$ are linearly independent.