Example (13): show that $e^{x}, e^{-x}, 2e^{x}$ linearly dependent in $(-\infty, \infty)$

Solution; from wronskian

$$w(e^{x}, e^{-x}, 2e^{x}) = \begin{vmatrix} e^{x} & e^{-x} & 2e^{x} \\ e^{x} & -e^{-x} & 2e^{x} \\ e^{x} & e^{-x} & 2e^{x} \end{vmatrix} = 0$$

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فالمحدد يساوي صفر

$\therefore$ the function $e^{x}, e^{-x}, 2e^{x}$ are linearly dependent functions in $(-\infty, \infty)$.

Theorem (7): If $g_1, g_2, \dots, g_n$ are solutions of the $L_n y = 0$ on the interval I which contain the point x_0 then,

$$w(g_1, g_2, \dots, g_n)(x) = \exp\left[-\int_{x_0}^{x} \frac{a_{n-1}(t)}{a_0(t)}\, dt\right] \cdot w(g_1, g_2, \dots, g_n)(x_0) \quad \text{----} (1)$$

Proof: we shall prove relation (1) when $n=2$, and if $n>2$ the proof will be in similar method

from definition of wronskian we get:

$$w(g_1, g_2)(x) = g_1(x) g_2'(x) - g_1'(x) g_2(x)$$

نشتق المعادلة

$$w'(g_1, g_2)(x) = g_1 g_2'' + g_1' g_2' - (g_1' g_2' + g_1'' g_2)$$

$$= g_1 g_2'' + \cancel{g_1' g_2'} - \cancel{g_1' g_2'} - g_1'' g_2$$

$$w'(g_1, g_2)(x) = g_1(x) g_2''(x) - g_1''(x) g_2(x) \quad \text{----} (2)$$

Since g_1, g_2 are solution of $L_2 y = 0$, then the following equations satisfies:

$$a_0(x) g_1''(x) + a_1(x) g_1'(x) + a_2(x) g_1(x) = 0$$

$$a_0(x) g_2''(x) + a_1(x) g_2'(x) + a_2(x) g_2(x) = 0 \qquad \forall x \in I$$

Since $\boxed{L_2 y = a_0 y'' + a_1 y' + a_2 y = 0}$

$$\Rightarrow \left.\begin{aligned} g_1''(x) &= -\frac{a_1(x)}{a_0(x)} g_1'(x) - \frac{a_2(x)}{a_0(x)} g_1(x) \\ g_2''(x) &= -\frac{a_1(x)}{a_0(x)} g_2'(x) - \frac{a_2(x)}{a_0(x)} g_2(x) \end{aligned}\right\} \cdots (2)$$

subsitituting in (1) we get:

$$w'(g_1, g_2)(x) = g_1\left(\frac{-a_1}{a_0} g_2' - \frac{a_2}{a_0} g_2\right) - g_2\left(\frac{-a_1}{a_0} g_1' - \frac{a_2}{a_0} g_1\right)$$

$$\Rightarrow w'(g_1, g_2)(x) = \frac{-a_1(x)}{a_0(x)} w(g_1, g_2)(x)$$

i.e. $$\frac{dw}{dx} = \frac{-a_1(x)}{a_0(x)} w(g_1, g_2)(x)$$ معادلة تفاضلية تنفصل متغيراتها

$$\frac{dw}{w} = \frac{-a_1(x)}{a_0(x)} dx$$

نكامل

$$\ln w = \int_{x_0}^{x} \frac{-a_1(t)}{a_0(t)} dt + \ln c$$

$$\ln w - \ln c = \int_{x_0}^{x} \frac{-a_1(t)}{a_0(t)} dt$$

$$\Rightarrow \ln \frac{w}{c} = -\int_{x_0}^{x} \frac{a_1(t)}{a_0(t)}\, dt$$

$$\Rightarrow \frac{w}{c} = \exp\left[-\int_{x_0}^{x} \frac{a_1(t)}{a_0(t)}\, dt\right]$$

$$\Rightarrow w(g_1, g_2)(x) = c \exp\left[-\int_{x_0}^{x} \frac{a_1(t)}{a_0(t)}\, dt\right] \cdots ③ \qquad \boxed{e^0 = 1}$$

if $x = x_0$ in ③, then $c = w(g_1, g_2)(x_0)$

$$\therefore w(g_1, g_2)(x) = \exp\left[-\int_{x_0}^{x} \frac{a_1(t)}{a_0(t)}\, dt\right] w(g_1, g_2)(x_0).$$

<u>Theorem (8):</u> If y_1 is a solution of the D.E. $L_2 y = 0$ on the interval I, such that $y_1(x) \neq 0$, then any solution of this D.E. in I satisfy the relation

$$y_2(x) = y_1(x) \int \frac{e^{-\int \frac{a_1(x)}{a_0(x)} dx}}{y_1^2(x)}\, dx$$

and both solutions are linearly independent in I, such that $w(y_1, y_2)(x_0) \neq 0$, where x_0 any point in I.

<u>Proof:</u> from theorem ⑦ we have

$$w(y_1, y_2)(x) = c \exp\left[-\int \frac{a_1(x)}{a_0(x)}\, dx\right]$$

$$y_1(x) y_2'(x) - y_2(x) y_1'(x) = c \exp\left[-\int \frac{a_1(x)}{a_0(x)}\, dx\right]$$

بالنسبة ل y_2

this is Linear D.E. of first order for y_2

dividing by $y_1^2(x)$ we get:

$$\frac{y_1(x)y_2'(x) - y_2(x)y_1'(x)}{y_1^2(x)} = \frac{c\exp\left[-\int \frac{a_1(x)}{a_0(x)}dx\right]}{y_1^2(x)}$$

$$d\left(\frac{y_2(x)}{y_1(x)}\right) = \frac{c\exp\left[-\int \frac{a_1(x)}{a_0(x)}dx\right]}{y_1^2(x)}$$

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$$\frac{y_2(x)}{y_1(x)} = \int \frac{c\exp\left[-\int \frac{a_1(x)}{a_0(x)}dx\right]}{y_1^2(x)}dx + k \quad , k\text{-constant}$$

$$\Rightarrow y_2(x) = c\,y_1(x)\int \frac{\exp\left[-\int \frac{a_1(x)}{a_0(x)}dx\right]}{y_1^2(x)}dx + k\,y_1(x)$$

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Since k, c arbirtrary constants, we choose $c=1$, $k=0$

then $$y_2(x) = y_1(x)\int \frac{\exp\left[-\int \frac{a_1(x)}{a_0(x)}dx\right]}{y_1^2(x)}dx$$

Example(14): Find wronskian of the solution for the D.E. $xy'' + y' + xy = 0$ which satisfies the following conditions $y_1(1) = 1$, $y_2(1) = 3$, $y_1'(1) = 2$, $y_2'(1) = 0$

Solution; we note $L_2y = a_0y'' + a_1y' + a_2y$

i.e. $a_0(x) = x$, $a_1(x) = 1$, $a_2(x) = x$

from theorem (7)

$$w(y_1, y_2)(x) = \exp\left[-\int_{x_0}^{x} \frac{a_1(t)}{a_0(t)}dt\right] w(y_1, y_2)(x_0)$$

$$= \exp\left[-\int_1^x \frac{1}{t}dt\right] w(y_1, y_2)(1)$$

$$w(y_1,y_2)(x) = \exp\left[-\int_1^x \frac{1}{t}\,dt\right]\cdot(0-6)$$

$$= \exp\left[-\int_1^x \frac{dt}{t}\right]\cdot(-6)$$

$$= (-6)\exp\left[-\ln x + \ln 1\right] \qquad \ln 1 = 0$$

$$w(y_1,y_2) = -6 \exp \ln x^{-1} = -\frac{6}{x}.$$

Example(15): Find wronskian of the solution for the D.E.

$y'' - (\sin x)y' + 3(\tan x)y = 0$, which satisfies the following

Conditions $y_1(0) = 1$, $y_2(0) = 2$, $y_1'(0) = 0$, $y_2'(0) = 1$

Solution: we note $a_0(x) = 1$, $a_1(x) = -\sin x$, $a_3(x) = 3\tan x$

$$w(y_1,y_2)(x) = \exp\left[-\int_{x_0}^x \frac{a_1(t)}{a_0(t)}\,dt\right] w(y_1,y_2)(x_0)$$

$$= \exp\left[-\int_0^x \frac{-\sin t}{1}\,dt\right] w(y_1,y_2)(0)$$

$$= \exp\left[-\int_0^x -\sin t\,dt\right]\cdot(1-0)$$

$$= \exp\left[-\cos x + \cos 0\right]$$

$$= e^{-\cos x}\cdot e^{1}$$

$$\therefore w(y_1,y_2)(x) \simeq 2.7\,e^{-\cos x}.$$

Example(16): Find the second solution of the D.E. $y''-4y'+4y=0$, where $y_1(x)=e^{2x}$, and prove that both solutions are linearly independent.

solution: from the relation in Th. (8)

$$y_2(x)=y_1(x)\int \frac{\exp\left[-\int \frac{a_1(x)}{a_0(x)}dx\right]}{y_1^2(x)}\,dx$$

$$=e^{2x}\int \frac{\exp\left[-\int -\frac{4}{1}\,dx\right]}{\left(e^{2x}\right)^2}\,dx$$

$$=e^{2x}\int \frac{e^{4x}}{e^{4x}}\,dx=e^{2x}\int dx=xe^{2x}=y_2(x).$$

to prove $y_1(x), y_2(x)$ linearly independent, we note

$$w\left(e^{2x}, xe^{2x}\right)=\begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x}+e^{2x} \end{vmatrix}=e^{4x}\neq 0$$

$\therefore$ the solutions are linearly independent.

Example (17): Find the second solution of the D.E. $(1-x^2)y''-2xy'+2y=0$, $y_1(x)=x$, $-1<x<1$, and prove both solutions are linearly independent.

solution: from the relation in th(8)

$$y_2(x)=y_1(x)\int \frac{\exp\left[-\int \frac{a_1(x)}{a_0(x)}dx\right]}{y_1^2(x)}\,dx$$