$$y_2(x) = x\int \frac{\exp\left[-\int \frac{-2x}{1-x^2}\,dx\right]}{x^2}\,dx$$

$$= x\int \frac{\exp\left[-\ln|1-x^2|\right]dx}{x^2} = x\int \frac{e^{-\ln|1-x^2|}}{x^2}\,dx$$

$$= x\int \frac{e^{\ln|1-x^2|^{-1}}}{x^2}\,dx = x\int \frac{dx}{x^2(1-x^2)}$$

$$= x\int\left[\frac{1}{x^2} + \frac{1}{2(x+1)} - \frac{1}{2(x-1)}\right]dx$$

$$= x\left[-\frac{1}{x} + \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x-1|\right]$$

$$y_2(x) = -1 + \frac{x}{2}\ln|x+1| - \frac{x}{2}\ln|x-1|$$

to prove $y_1(x), y_2(x)$ linearly indp.

نكامل المقدار بتجزئة الكسور

$$\frac{1}{x^2(1-x^2)} = \frac{1}{-x^2(x^2-1)}$$

$$= \frac{1}{-x^2(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{x-1}$$

بعد حل المعادلات نجد

$A = 0$, $B = 1$, $C = \frac{1}{2}$, $D = \frac{-1}{2}$

$$\therefore \frac{1}{x^2(1-x^2)} = \frac{1}{x^2} + \frac{1/2}{x+1} + \frac{(-1/2)}{x-1}$$

$$W(y_1, y_2) = \begin{vmatrix} x & -1 + \frac{x}{2}\ln|x+1| - \frac{x}{2}\ln|x-1| \\ 1 & \frac{x}{2x+2} + \frac{1}{2}\ln|x+1| - \frac{x}{2x-2} - \frac{\ln|x-1|}{2} \end{vmatrix} = \frac{-1}{x^2-1} \neq 0$$

Since $-1 < x < 1$

Exercises

(1) Prove that the following functions are linearly independent on I

a. $1, e^{-x}, 2e^{2x}$, $I = (-\infty, \infty)$

b. $x\ln x$, $\ln x$ $\qquad I=(0,\infty)$

c. $1, \sin^2 x, 1-\cos x$ $\qquad I=(-\infty,\infty)$

d. $x, \sqrt{1-x^2}$ $\qquad I=(-1,1)$

2. Find wronskian of the solution for the following D.Es.

a. $x^2y''+xy'+(x^2+1)y=0$
with $y_2(1)=1$, $y_2'(1)=1$, $y_1'(1)=0$, $y_1(1)=0$

b. $x^2y''-3xy'+y=0$
with $y_1(0)=1$, $y_1'(0)=0$, $y_2'(0)=1$, $y_2(0)=0$

3. Find the second solution for the following D.Es. and prove that these solutions are linearly independent

a. $3xy''-y'=0$, $y_1(x)=1$

b. $y''+(\tan x)y'-6(\cot^2 x)y=0$, $y_1(x)=\sin^3 x$

Linear Differential Equations of n-th order with constant coefficients

These D.Es has the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = f(x) \quad \cdots (1)$$

where $a_0, a_1, a_2, \cdots, a_n$ constants

we can solve D.E. (1) by many methods

المؤثر
1. Method of operator.

معاملات غير المحددة
2. Method of undetermind coefficients.

تغير الثوابت
3. Method of variation of parameters.

Method of variation of parameters.

It is general method to find particular solution of non-homogenous D.E. In this method we change the arbitrary constants c_i in general solution $\sum_{i=1}^{n} c_i y_i$ of homogenous D.E. $L_n y = 0$ by functions v_i such that $\sum_{i=1}^{n} v_i y_i$ become particular solution of non-homogenous D.E. $L_n y = h(x)$, hence we called this method the method of

طريقة تغير الثوابت (البارومترات)
variation of parameters.

Theorem (9): If g_1, g_2 are linearly independent solutions of D.E. $L_2 y = 0$, on I. then we can evaluate particular solution y_p of D.E. $L_2 y = h(x)$ from the relation :

تابع

$$y_p(x) = \int_{x_0}^{x} \frac{[g_1(t) g_2(x) - g_1(x) g_2(t)]}{a_0 W(g_1, g_2)(t)} \cdot h(t)\, dt \quad --- (1)$$

where $x_0 \in I$.

(C تقبل الاشتقاق مرة واحدة في الفترة I)

Proof: Let v_1, v_2 be functions in $C^{(1)}(I)$ such that $\forall x \in I$

$$y_p(x) = v_1(x) g_1(x) + v_2(x) g_2(x) \quad --- (2) \quad \text{and } \forall x \in I$$

$$v_1'(x) g_1(x) + v_2'(x) g_2(x) = 0 \quad --- (3)$$

Since $y_p(x)$ is particular solution of the D.E. $L_2 y = h(x)$

then $a_0 y_p''(x) + a_1 y_p'(x) + a_2 y_p(x) = h(x) \quad --- (4)$

from (2) $\quad y_p'(x) = v_1' g_1 + v_1 g_1' + v_2' g_2 + v_2 g_2'$

from (3)

$$y_p'(x) = v_1(x) g_1'(x) + v_2(x) g_2'(x)$$

$$y_p''(x) = v_1(x) g_1''(x) + v_1'(x) g_1'(x) + v_2(x) g_2''(x) + v_2'(x) g_2'(x)$$

substituting y_p, y_p', y_p'' in (4) we get:

$$a_0 [v_1 g_1'' + v_1' g_1' + v_2 g_2'' + v_2' g_2'] + a_1 [v_1 g_1' + v_2 g_2'] +$$

$$+ a_2 [v_1 g_1 + v_2 g_2] = h(x)$$

$$\Rightarrow a_0[v_1'g_1' + v_2'g_2'] = h(x) \quad \cdots ⑤$$

from ③ and ⑤ we get

$$v_1'g_1 + v_2'g_2 = 0 \quad \cdots ③$$

$$a_0v_1'g_1' + a_0v_2'g_2' = h(x) \quad \cdots ⑤$$

by using Grammer rule we get:

$$v_1' = \frac{\begin{vmatrix} 0 & g_2 \\ h & a_0g_2' \end{vmatrix}}{\begin{vmatrix} g_1 & g_2 \\ a_0g_1' & a_0g_2' \end{vmatrix}} = \frac{-hg_2}{a_0g_1g_2' - a_0g_1'g_2} = \frac{-hg_2}{a_0[g_1g_2' - g_1'g_2]}$$

$$\Rightarrow v_1' = \frac{-hg_2}{a_0 W(g_1, g_2)(x)}$$

also

$$v_2' = \frac{\begin{vmatrix} g_1 & 0 \\ a_0g_1' & h \end{vmatrix}}{\begin{vmatrix} g_1 & g_2 \\ a_0g_1' & a_0g_2' \end{vmatrix}} = \frac{hg_1}{a_0g_1g_2' - a_0g_1'g_2} = \frac{hg_1}{a_0W(g_1, g_2)(x)}$$

تكامل

$$v_1 = -\int_{x_0}^{x} \frac{g_2(t)h(t)}{a_0W(g_1,g_2)(t)}\,dt \qquad v_2 = \int_{x_0}^{x} \frac{g_1(t)h(t)}{a_0W(g_1,g_2)(t)}\,dt$$

subsitituting in ②, we get the required relation