$$f(x,y) = x^3 y + y^3 + x^2 + c$$

ونلاحظ ان الناتج هو نفسه في الطريقة الاولى .

Example(24): Solve the D.E.

$$(3x^2 + 3xy^2)dx + (3x^2y - 3y^2 + 2y)dy = 0 \quad \dots (1)$$

Solution: $M(x,y) = 3x^2 + 3xy^2$, $N(x,y) = 3x^2y - 3y^2 + 2y$

$$\frac{\partial M}{\partial y} = 6xy \quad , \quad \frac{\partial N}{\partial x} = 6xy$$

we note $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ so the D.E. is exact.

thus there exist a function $f(x,y)$ such that

$$f(x,y) = \int M(x,y)\,dx + \phi(y)$$

$$= \int (3x^2 + 3xy^2)\,dx + \phi(y) \qquad \text{نكامل}$$

$$f(x,y) = x^3 + \frac{3}{2}x^2y^2 + \phi(y) \quad \dots (2)$$

نشتق (2) بالنسبة لـ y

$$\frac{\partial f}{\partial y} = 3x^2y + \phi'(y)$$

وبما ان $\frac{\partial f}{\partial y} = N(x,y)$ فإنه

$$3x^2y + \phi'(y) = 3x^2y - 3y^2 + 2y$$

$$\Rightarrow \phi'(y) = -3y^2 + 2y$$

نكامل

$$\Rightarrow \phi(y) = -y^3 + y^2 + c$$

نعوض في المعادلة (2)

$$\therefore f(x,y) = x^3 + \frac{3}{2}x^2y^2 - y^3 + y^2 + c.$$

Example (25): Solve the D.E.

$$\left(3x^2 + 2\frac{y}{x}\right)dx + \left(2\ln 3x + \frac{3}{y}\right)dy = 0 \quad , x > 0 , y \neq 0 \quad \text{--- (1)}$$

Solution: $M(x,y) = 3x^2 + 2\frac{y}{x}$, $N(x,y) = 2\ln 3x + \frac{3}{y}$

$$\frac{\partial M}{\partial y} = \frac{2}{x} , \quad \frac{\partial N}{\partial x} \; 2 \cdot \frac{3}{3x} = \frac{2}{x}$$

$\because$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \therefore \text{ the D.E. exact.}$$

thus there exist a function $f(x,y)$ such that

$$f(x,y) = \int M(x,y)\,dx + \phi(y)$$

$$= \int \left(3x^2 + 2\frac{y}{x}\right)dx + \phi(y) \qquad \text{نكامل}$$

$$\therefore f(x,y) = x^3 + 2y\ln x + \phi(y) \quad \text{---- (2)}$$

نشتق المعادلة (2) بالنسبة لـ y

$$\frac{\partial f}{\partial y} = 2\ln x + \phi'(y)$$

بما أن $\frac{\partial f}{\partial y} = N(x,y)$ فإنه

$$2\ln x + \phi'(y) = 2\ln 3x + \frac{3}{y}$$

نكامل نحصل على

$$\phi(y) = 2y\ln 3x + 3\ln y - 2y\ln x + C$$

نعوض في المعادلة (2) فنجد

$$f(x,y) = x^3 + 2y\ln x + 2y\ln 3x + 3\ln y - 2y\ln x + C$$

is the solution

<u>Example (26)</u>: prove that the following equation exact, then find the general solution of it.

$$(x^2-y)\,dx+(y^2-x)\,dy=0 \quad \ldots(1)$$

<u>Solution</u>:

$$M(x,y)=x^2-y \quad , \quad N(x,y)=y^2-x$$

$$\frac{\partial M}{\partial y}=-1 \quad , \quad \frac{\partial N}{\partial x}=-1$$

$\because \dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$, So the D.E. exact

thus there exist a function $f(x,y)$ s.t.

$$f(x,y)=\int M(x,y)\,dx+\phi(y)$$

$$=\int (x^2-y)\,d_x+\phi(y)$$

$$=\frac{x^3}{3}-xy+\phi(y) \quad \ldots(2)$$

تشتق بالنسبة لـ y

$$\frac{\partial f}{\partial y}=-x+\phi'(y)$$

$\dfrac{\partial f}{\partial y}=N(x,y)$ عادل

$$\cancel{-x}+\phi'(y)=y^2\cancel{-x}$$

$$\therefore \phi'(y)=y^2$$

نكامل بالنسبة لـ y

$$\therefore \phi(y)=\frac{y^3}{3}+c$$

$\therefore f(x,y)=\dfrac{x^3}{3}-xy+\dfrac{y^3}{3}+c$ is the general sol.

ملاحظة: لحل المعادلات التفاضلية التامة نتبع احدى الطريقتين

الاولى:

نفرض ان $f(x,y) = \int M(x,y)\,dx + \phi(y)$ نكامل بالنسبة لـ x ثم نشتق الناتج بالنسبة لـ y ونساويه بـ $N(x,y)$ ثم نكامل الناتج لايجاد $\phi(y)$

الثانية:

نفرض ان $f(x,y) = \int N(x,y)\,dy + g(x)$ نكامل بالنسبة لـ y ثم نشتق الناتج بالنسبة لـ x ونساويه بـ $M(x,y)$ ثم نكامل الناتج لايجاد $g(x)$.

Exercises: prove that the following D.E. exact and solve it.

1. $e^{3x}(dy + 2y\,dx) = x^2\,dx$
2. $y^3 \sin 2x\,dx - 3y^2 \cos^2 x\,dy$, $\cos^2 x = \dfrac{1+\cos 2x}{2}$
3. $2\dfrac{x}{y}\,dy + \left(2\ln 5y + \dfrac{1}{x}\right)dx = 0$
4. $\dfrac{y}{x}\,dy - \left(\dfrac{y^2}{2x^2} + x\right)dx = 0$
5. $3y(x^2-1)\,dx + (x^3 + 8y - 3x)\,dy = 0$, $y(0) = 1$
6. $(xy^2 + x - 2y + 3)\,dx + x^2y\,dy = 2(x+y)\,dy$, $y(1) = 1$

Integral factor عامل تكامل

In the D.E. $M(x,y)\,dx + N(x,y)\,dy = 0$ ---(1)

if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then D.E. (1) called not exact, غير تامة

then we transfor D.E. (1) to exact by multiplying the equation by the function called integral factor which denoted by (u)

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let

$$h(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \quad \text{or} \quad g(y) = -\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M}$$

then

$$u = e^{\int h(x)\,dx} \quad \text{or} \quad u = e^{\int g(y)\,dy}$$

Example (27): Solve the D.E.

$$(2y - 4x^2)\,dx + x\,dy = 0 \quad \text{---} (1)$$

solution: $M(x,y) = 2y - 4x^2$, $N(x,y) = x$

$$\frac{\partial M}{\partial y} = 2 , \quad \frac{\partial N}{\partial x} = 1 \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ D.E. (1) not exact

$$h(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2-1}{x} = \frac{1}{x}$$

عامل التكامل
$$\therefore u = e^{\int h(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

بضرب المعادلة (1) في عامل التكامل نحصل على

$$(2xy - 4x^3)dx + x^2 dy = 0 \quad --- (2)$$

Now $M_1(x,y) = 2xy - 4x^3$, $N_1(x,y) = x^2$

$$\frac{\partial M_1}{\partial y} = 2x \; , \; \frac{\partial N_1}{\partial x} = 2x \Rightarrow \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

تامة

$\therefore$ D.E. (2) exact.

$\therefore$ there exist a function $f(x,y)$ such that

$$f(x,y) = \int M_1(x,y)dx + \phi(y)$$
$$= \int (2xy - 4x^3)dx + \phi(y)$$
$$f(x,y) = x^2y - x^4 + \phi(y) \quad --- (3)$$

نشتق (3) بالنسبة لـ y

$$\frac{\partial f}{\partial y} = x^2 + \phi'(y)$$

بما أن $\frac{\partial f}{\partial y} = N_1(x,y)$

$$x^2 + \phi'(y) = x^2 \Rightarrow \phi'(y) = 0 \xRightarrow{\text{نكامل}} \phi(y) = c$$

نعوضها في المعادلة (3) نحصل

$$f(x,y) = x^2y - x^4 + c .$$

Example (28) : Solve the D.E.

$$(y^4 + x^3)dx + 8xy^3dy = 0 \quad --- (1)$$

<u>solution:</u> $M(x,y) = y^4 + x^3$, $N(x,y) = 8xy^3$

$$\frac{\partial M}{\partial y} = 4y^3 \ , \ \frac{\partial N}{\partial x} = 8y^3 \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

غير تامة

$\therefore$ D.E. (1) not exact

$$h(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{4y^3 - 8y^3}{8xy^3} = \frac{-4y^3}{8xy^3} = \frac{-1}{2x}$$

$$\therefore u = e^{\int h(x)\,dx} = e^{\int \frac{-1}{2x}dx} = e^{-\frac{1}{2}\ln x} = e^{\ln x^{-1/2}} = x^{-1/2} = \frac{1}{\sqrt{x}}$$

نضرب المعادلة التفاضلية (1) في عامل التكامل فنحصل على

$$\left(\frac{1}{\sqrt{x}}y^4 + x^3\frac{1}{\sqrt{x}}\right)dx + 8xy^3\frac{1}{\sqrt{x}}dy = 0$$

$$\left(\frac{1}{\sqrt{x}}y^4 + x^{5/2}\right)dx + 8\sqrt{x}\,y^3dy = 0 \quad \text{---}(2)$$

$$M_1(x,y) = \frac{1}{\sqrt{x}}y^4 + x^{5/2} \ , \ N_1(x,y) = 8\sqrt{x}\,y^3$$

$$\frac{\partial M_1}{\partial y} = \frac{4}{\sqrt{x}}y^3 \ , \ \frac{\partial N_1}{\partial x} = \frac{4}{\sqrt{x}}y^3$$

$\therefore$ D.E. (2) is exact

$\therefore$ there exist a function $f(x,y)$ such that

$$f(x,y) = \int M_1(x,y)\,dx + \phi(y)$$

$$= \int\left(\frac{y^4}{\sqrt{x}} + x^{5/2}\right)dx + \phi(y)$$

نكامل

$$f(x,y) = y^4 \cdot \frac{x^{1/2}}{1/2} + \frac{x^{7/2}}{7/2} + \phi(y)$$

$$f(x,y) = 2y^4\sqrt{x} + \frac{2}{7}x^{7/2} + \phi(y) \quad --- (4)$$

بما ان $\frac{\partial f}{\partial y} = N_1(x,y)$

$$\frac{\partial f}{\partial y} = 8y^3\sqrt{x} + \phi'(y) \Rightarrow 8y^3\sqrt{x} + \phi'(y) = 8y^3\sqrt{x}$$

$$\Rightarrow \phi'(y) = 0 \Rightarrow \phi(y) = C$$

نعوض في المعادلة (4) نحصل على

$$f(x,y) = 2y^4\sqrt{x} + \frac{2}{7}x^{7/2} + C$$

Other Kinds of non_exact D.E.

There are other kinds of non-exact D.E. as in the following examples:

Example (29): Solve the D.Es.

1. $x\,dy + y\,dx = xy^3 dx$

Solution: $d(xy) = xy^3 dx$

نكامل

$$xy = \frac{x^2}{2}y^3 + C \Rightarrow y = \frac{x}{2}y^3 + \frac{C}{x}$$

2. $x\,dy - y\,dx = (x^2 + xy - 2y^2)dx$

Solution:

بالقسمة على x^2 نجد

$$\frac{x\,dy - y\,dx}{x^2} = \left(1 + \frac{y}{x} - 2\frac{y^2}{x^2}\right)dx$$

$$d\left(\frac{y}{x}\right) = \left(1 + \frac{y}{x} - 2\frac{y^2}{x^2}\right)dx$$

تكامل فقص على

$$\frac{y}{x} = x + y\ln|x| + \frac{2y^2}{x} + c$$

3. $e^{x^2} dy + 2xy\,e^{x^2} dx = x\,dx$

<u>solution;</u> $d(e^{x^2} y) = x\,dx$

تكامل

$$e^{x^2} y = \frac{x^2}{2} + c$$

$$\Rightarrow y = \frac{x^2}{2} e^{-x^2} + c\,e^{-x^2}$$

4. $y\,dx - x\,dy = x\,dx$

<u>solution:</u>

بالقسمة على y^2 نجد

$$\frac{y\,dx - x\,dy}{y^2} = \frac{x}{y^2} dx$$

$$d\left(\frac{x}{y}\right) = \frac{x}{y^2} dx$$

تكامل فقص على

$$\frac{x}{y} = \frac{x^2}{2} \cdot \frac{1}{y^2} + c$$

$$x = \frac{x^2}{2} \cdot \frac{1}{y} + c\,y^{-1}$$

Exercises : Solve the following

1. $x\,dx + y\,dy + (x^2+y^2)^2 dx = 0$

2. $(4y + 3x^2)dy + 2xy\,dx = 0$

3. $(2\cos y + 4x^2)dx - x\sin y\,dy = 0$