Theorem (10): If $g_1, g_2, \dots, g_n$ Linearly independent Solution of D.E. $L_n y = h(x)$ on I, then

$$y_p(x) = \sum_{k=1}^{n} g_k(x) \cdot \int_{x_0}^{x} \frac{W_k(t) \cdot h(t)}{a_0\, W(g_1, g_2, \dots, g_n)(t)}\, dt$$

where W_k is determinant (محدد), which obtain from wronskian by change the column $(g_k, g_k', g_k'', \dots, g_k^{(n-1)})$ with the column $(0, 0, \dots, 0, 1)$.

Example (18): By the method of variation of parameters Solve the D.E. $y'' - 3y' - 10y = e^{x}$, $-\infty < x < \infty$

Solution: $K^2 - 3K - 10 = e^{x}$ المعادلة المميزة

$(K-5)(K+2) = 0 \Rightarrow K_1 = 5, K_2 = -2$ الجذور الحقيقية مختلفة

$\therefore \; y_c(x) = c_1 e^{5x} + c_2 e^{-2x}$ الحل المتمم

لكي نحل بطريقة تغيير الثوابت (Theorem 9) ، من الحل المتمم نجد أن

$$g_1(x) = e^{5x} , \quad g_2(x) = e^{-2x}$$

from (Theorem (9))

$$y_p(x) = \int_0^x \frac{e^{-2x} \cdot e^{5t} - e^{-2t} \cdot e^{5x}}{1 \cdot W(e^{5t}, e^{-2t})} \cdot e^{t}\, dt$$

we note

$$W(e^{5t}, e^{-2t}) = \begin{vmatrix} e^{5t} & e^{-2t} \\ 5e^{5t} & -2e^{-2t} \end{vmatrix} = -2e^{3t} - 5e^{3t} = -7e^{3t}$$

$$y_p(x) = \int_0^x \left[\frac{e^{-2x} \cdot e^{5t} - e^{-2t} \cdot e^{5x}}{-7e^{3t}} \right] e^{t} \, dt$$

$$y_p(x) = \int_0^x \left[\frac{e^{-2x} \cdot e^{6t} - e^{-t} \cdot e^{5t}}{-7e^{3t}} \right] dt$$

$$= \int_0^x \left[\frac{-e^{-2x}}{7} e^{3t} + \frac{e^{-4t}}{7} e^{5x} \right] dt$$

$$= \left[\frac{e^{-2x}}{-7} \cdot \frac{e^{3t}}{3} + \frac{e^{5x}}{7} \cdot \frac{e^{-4t}}{-4} \right]_0^x$$

$$y_p(x) = \left[\frac{e^{-2x}}{-21} e^{3x} + \frac{e^{5x}}{-28} e^{-4x} \right] - \left[\frac{e^{-2x}}{-21} \cdot e^{0} + \frac{e^{5x}}{-28} \cdot e^{0} \right]$$

$$= \frac{e^{x}}{-21} - \frac{e^{x}}{28} + \frac{e^{-2x}}{21} + \frac{e^{5x}}{28}$$

عبارات الحدين $\frac{e^{5x}}{98} + \frac{e^{-2x}}{21}$ قد ظهرا في الحل المتمم لهذا نهمل منه الحل الخاص

$$y_p(x) = \frac{e^{x}}{21} - \frac{e^{x}}{28} = \frac{-7e^{x}}{84} = \frac{-1}{12} e^{x}$$

الحل العام

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1 e^{5x} + c_2 e^{-2x} - \frac{1}{12} e^{x} .$$

Example(19): By the method of variation of parameters solve the D.E. $y''' - 4y' = \sin x$

المعادلة المميزة

Solution: the characterstic eq. of the D.E. is

$$K^3 - 4K = 0 \implies K(K^2 - 4) = 0$$

$\Rightarrow K(K-2)(K+2)=0 \Rightarrow k_1=0,\ k_2=2,\ k_3=-2$

جذور حقيقية مختلفة

$\therefore\ y_c(x) = c_1 e^{0.x} + c_2 e^{2x} + c_3 e^{-2x}$

$y_c(x) = c_1 + c_2 e^{2x} + c_3 e^{-2x}$ الحل المتمم

الحل الخاص

to find the particular solution, we note :

$g_1(x) = e^0 = 1 \quad , \quad g_2(x) = e^{2x} \quad , \quad g_3(x) = e^{-2x}$

$$W(g_1,g_2,g_3)(t) = \left|\begin{matrix} 1 & e^{2t} & e^{-2t} \\ 0 & 2e^{2t} & -2e^{-2t} \\ 0 & 4e^{2t} & 4e^{-2t} \end{matrix}\right| \begin{matrix} 1 & e^{2t} \\ 0 & 2e^{2t} \\ 0 & 4e^{2t} \end{matrix}$$

$$= \left[8e^{0.t} + 0 + 0\right] - \left[0 - 8e^{0.t} + 0\right] = 8+8 = 16$$

$$W_1(t) = \left|\begin{matrix} 0 & e^{2t} & e^{-2t} \\ 0 & 2e^{2t} & -2e^{-2t} \\ 1 & 4e^{2t} & 4e^{-2t} \end{matrix}\right| = -4$$

$$W_2(t) = \left|\begin{matrix} 1 & 0 & e^{-2t} \\ 0 & 0 & -2e^{-2t} \\ 0 & 1 & 4e^{-2t} \end{matrix}\right| = 2e^{-2t}$$

$$W_3(t) = \left|\begin{matrix} 1 & e^{2t} & 0 \\ 0 & 2e^{2t} & 0 \\ 0 & 4e^{2t} & 1 \end{matrix}\right| = 2e^{2t}$$

From. (theorem (10))

$$y_p(x) = \sum_{k=1}^{n} g_k(x) \int_{x_0}^{x} \frac{w_k(t) \cdot h(t)}{a_0 W(g_1, g_2, \ldots, g_n)(t)} \, dt$$

$$y_p(x) = g_1(x) \int_0^x \frac{w_1(t) \cdot h(t)}{a_0 W(g_1, g_2, g_3)(t)} \, dt + g_2(x) \int_0^x \frac{w_2(t) \cdot h(t)}{a_0 W(g_1, g_2, g_3)} \, dt$$

$$+ g_3(x) \int_0^x \frac{w_3(t) \cdot h(t)}{a_0 W(g_1, g_2, g_3)(t)} \, dt.$$

$$y_p(x) = 1 \cdot \int_0^x \frac{-4 \sin t}{16} \, dt + e^{2x} \int_0^x \frac{2 e^{-2t} \cdot \sin t}{16} \, dt$$

$$+ e^{-2x} \int_0^x \frac{2 e^{2t} \cdot \sin t}{16} \, dt$$

$$y_p(x) = \frac{1}{4} \cos t \Big]_0^x + \frac{e^{2x}}{8} \left[\frac{e^{-2t}(-2\sin t - \cos t)}{(-2)^2 + 1^2} \right]_0^x + \frac{e^{-2x}}{8} \left[\frac{e^{2t}(2\sin t - \cos t)}{5} \right]_0^x$$

$$y(x) = \frac{1}{4} \cos x - \frac{1}{4} + \frac{1}{40}(-2\sin x - \cos x) + \frac{e^{2x}}{40} + \frac{1}{40}(2\sin x - \cos x)$$

$$+ \frac{e^{-2x}}{40}$$

باهمال الحدين $\frac{1}{4}$ ، $\frac{e^{2x}}{40}$ ، $\frac{e^{-2x}}{40}$ لانها ظهرت في الحل المتمم كثوابت

$$y_p(x) = \frac{1}{4} \cos x - \frac{1}{20} \cos x = \frac{1}{5} \cos x$$

$$\therefore \; y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1 + c_2 e^{2x} + c_3 e^{-2x} + \frac{1}{5} \cos x$$

$$\int e^{kx}\cdot \sin\lambda x\,dx = \frac{e^{kx}(k\sin\lambda x - \lambda\cos\lambda x)}{k^2+\lambda^2}$$

$$\int e^{kx}\cdot \cos\lambda x\,dx = \frac{e^{kx}(k\cos\lambda x + \lambda\sin\lambda x)}{k^2+\lambda^2}$$

<u>Example (20):</u> By the method of variation of parameters
Solve the D.E. $y''' - 7y' + 6y = 2\sin x$

<u>Solution;</u> the characteristic eq.

$k^3 - 7k + 6 = 0$

$(k^2+k-6)(k-1) = 0$

$(k+3)(k-2)(k-1) = 0$

$k_1 = 1,\ k_2 = -3,\ k_3 = 2$

$y_c(x) = c_1 e^{x} + c_2 e^{-3x} + c_3 e^{2x}$

$$\begin{array}{r|l} & k^2+k-6 \\ \hline k-1 & k^3-7k+6 \\ \text{بالطرح} & \mp k^3 \pm k^2 \\ \hline & k^2-7k+6 \\ & \mp k^2 \pm k \\ \hline & -6k+6 \\ & \pm 6k \mp 6 \\ \hline & /// \end{array}$$

hence $g_1(x) = e^{x}$, $g_2(x) = e^{-3x}$, $g_3(x) = e^{2x}$

$$W(e^{t}, e^{-3t}, e^{2t}) = \begin{vmatrix} e^{t} & e^{-3t} & e^{2t} \\ e^{t} & -3e^{-3t} & 2e^{2t} \\ e^{t} & 9e^{-3t} & 4e^{2t} \end{vmatrix} = -20$$

$$W_1(t) = \begin{vmatrix} 0 & e^{-3t} & e^{2t} \\ 0 & -3e^{-3t} & 2e^{2t} \\ 1 & 9e^{-3t} & 4e^{2t} \end{vmatrix} = 5e^{-t}$$

$$W_2(t) = \begin{vmatrix} e^{t} & 0 & e^{2t} \\ e^{t} & 0 & 2e^{2t} \\ e^{t} & 1 & 4e^{2t} \end{vmatrix} = -e^{3t}$$

$$W_3(t) = \begin{vmatrix} e^{t} & e^{-3t} & 0 \\ e^{t} & -3e^{-3t} & 0 \\ e^{t} & 9e^{-3t} & 1 \end{vmatrix} = -4e^{-2t}$$

From theorem (10) we find

$$y_p(x) = \sum_{k=1}^{n} g_k(x) \int_{x_0}^{x} \frac{W_k(t) \cdot h(t)}{a_0 W(g_1, g_2, \dots, g_n)(t)} \, dt$$

$$y_p(x) = e^{x} \int_0^x \frac{5e^{-t} \cdot 2\sin t}{-20} \, dt + e^{-3x} \int_0^x \frac{-e^{3t} \cdot 2\sin t}{-20} \, dt$$

$$+ e^{2x} \int_0^x \frac{-4e^{-2t} \cdot 2\sin t}{-20} \, dt.$$

$$y_p(x) = -\frac{e^{x}}{2} \int_0^x e^{-t} \sin t \, dt + \frac{e^{-3x}}{10} \int_0^x e^{3t} \sin t \, dt$$

$$+ \frac{2}{5} e^{2x} \int_0^x e^{-2t} \sin t \, dt.$$

$$y_p(x) = -\frac{e^{x}}{2} \left[\frac{-e^{-x}(3\sin x - e^{-x}\cos x}{2} + \frac{1}{2} \right] - \frac{e^{-3x}}{10} \left[\frac{e^{3x}(3\sin x - \cos x)}{10} + \frac{1}{10} \right]$$