Example (22): Find eigen values and eigen vectors for the matrix

$$A = \begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix}$$

solution; the characterstic eq. for A is:

$$P_2(r) = \begin{vmatrix} 3-r & 2 \\ -5 & 1-r \end{vmatrix} = r^2 - 4r + 13 = 0$$

the roots $r_1 = 2+3i$, $r_2 = 2-3i$. the eigen values, Subsitituting $r_1 = 2+3i$ in eq. (5), we get:

$(1-3i)v_{11} + 2v_{12} = 0$

$(-5)v_{11} + (-1-3i)v_{12} = 0$

then if $v_{11} = 2$, then $v_{12} = -1+3i$ and the eigen vector for $r_1 = 2+3i$ is $v_1 = \begin{bmatrix} 2 \\ -1+3i \end{bmatrix}$

in the Same method we find $v_2 = \begin{bmatrix} 2 \\ -1-3i \end{bmatrix}$ the eigen vector for $r_2 = 2-3i$

Solution for homogenous D.S.

The Solution of homogenous D.S. $Y' = AY$ depending on the eigen values of the matrix A, we have two cases

1. Distinst eigen values
2. repeated eigen values

1. Distinct eigen values القيم الذاتية (الجذور المميزة) المختلفة

Theorem (11):

1. For any eigen vector $\underline{v}$ corresponding to the eigen value $\underline{r}$ for the matrix A. the solution of homogenous D.S. $Y' = AY$ has the form $Y = ve^{rx}$

2. The solutions of D.S. $Y' = AY$ are linearly independent on the interval $(-\infty, \infty)$

3. The general solution of the D.S. has the form $\sum_{i=1}^{n} c_i v_i e^{r_i x}$

(تركيب خطي من الحلول)

Example (23): Solve the D.S.

$$Y' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} Y$$

ملاحظة: يمكن ان نكتب النظام بالشكل

$$y_1' = -y_2$$
$$y_2' = y_1$$

solution: the characteristic eq. for D.S. $p_2(r) = \begin{vmatrix} -r & -1 \\ 1 & -r \end{vmatrix} \Rightarrow p_2(r) = r^2 + 1 = 0$

$\therefore r_1 = i \ , \ r_2 = -i$

to find the eigen vector for $r_1 = i$, substituting $r_1 = i$ in eq. (5), we get:

$$(A - r_1 E)\underline{v_1} = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} -iv_{11} - v_{12} = 0 \\ v_{11} - iv_{12} = 0 \end{array}\right\} \Rightarrow v_{12} = -iv_{11}$$

Let $v_{11} = 1$, then $v_{12} = -i$ and $v_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ for $r_1 = i$ المتجه الخاص المقابل لـ

to find eigen vector for $r_2 = -i$, from (5) we get

$$(A - r_2 E)v_2 = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} iv_{21} - v_{22} = 0 \\ v_{21} + iv_{22} = 0 \end{array}\right\} \Rightarrow v_{22} = iv_{21}$$

Let $v_{21} = 1 \Rightarrow v_{22} = i$ and $v_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ for $r_2 = -i$

$\therefore$ The general solution is

$$y(x) = c_1 v_1 e^{r_1 x} + c_2 v_2 e^{r_2 x}$$

$$= c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{ix} + c_2 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{-ix} = \begin{bmatrix} c_1 e^{ix} + c_2 e^{-ix} \\ -c_1 i e^{ix} + c_2 i e^{-x} \end{bmatrix}$$

ملاحظة : من المثال السابق نجد انه اذا كانت $r = a + ib$ ، $b \neq 0$ قيمة خاصة للمصفوفة A وكان v متجها خاصا مقابل لـ r فيكون المتجه $\bar{v}$ (مرافق v) مقابلاً للقيمة الخاصة المرافقة $r = a - ib$.

Example (24): Solve the D.S.

$$\frac{dx}{dt} = -x + 8y$$

$$\frac{dy}{dt} = x + y$$

Solution: the characteristic eq. for D.S.

$$P_2(r) = \begin{vmatrix} -1-r & 8 \\ 1 & 1-r \end{vmatrix} = 0 \Rightarrow (-1-r)(1-r) - 8 = 0$$

$$\Rightarrow r^2 - 9 = 0 \Rightarrow r_1 = 3 \,,\ r_2 = -3$$ حقيقية مختلفة

to find the eigen vector for $r_1 = 3$, subsitituting $r_1 = 3$ in eq.(5), we get

$$(A - r_1 E)v_1 = \begin{bmatrix} -1-3 & 8 \\ 1 & 1-3 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -4 & 8 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} -4v_{11} + 8v_{12} &= 0 \\ v_{11} - 2v_{12} &= 0 \end{aligned}\right\} \Rightarrow v_{11} = 2v_{12}$$

Let $v_{12} = 1 \Rightarrow v_{11} = 2$, then $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for $r_1 = 3$

also to find the eigen vector for $r_2 = -3$, subsitituting $r_2 = -3$ in eq.(5), we get

$$(A - r_2 E)v_2 = \begin{bmatrix} -1+3 & 8 \\ 1 & 1+3 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 8 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} 2v_{21} + 8v_{22} &= 0 \\ v_{21} + 4v_{22} &= 0 \end{aligned}\right\} \Rightarrow v_{21} = -4v_{22}$$

Let $v_{22}=1 \Rightarrow v_{21}=-4$, $v_2=\begin{bmatrix}-4\\1\end{bmatrix}$ for $r_2=-3$

thus the particular solution

$$x_1=2e^{3t} \;,\; y_1=e^{3t}$$

$$x_2=-4e^{-3t} \;,\; y_2=e^{-3t}$$

the general solution

$$x=c_1x_1+c_2x_2$$

$$y=c_1y_1+c_2y_2$$

$$\Rightarrow \quad x=2c_1e^{3t}-4c_2e^{-3t}$$

$$y=c_1e^{3t}+c_2e^{-3t}$$

2. Repeated eigen values القيم الذاتية (الذاتية) مكررة

Theorem (12): Let r_0 be m-tiems repeated roots for the characteristic equation $P_n(r)=0$ for the matrix A, then the homogenous system $Y'=AY$ has the following solutions

$$Y_1(x)=B_{11}e^{r_0x}$$

$$Y_2(x)=B_{21}xe^{r_0x}+B_{22}e^{r_0x}$$

$$\vdots$$

$$Y_m(x)=B_{m1}x^{m-1}e^{r_0x}+\cdots+B_{mm}e^{r_0x}$$